Generalized Additive Games

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Cooperative TU-games

TU-game

A pair (N, v):

- N is the set of players;
- $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$.

A game with n players specifies a vector of $2^n - 1$ numbers.

Compact representation of TU-games

Several models from the literature focus on interaction situations which are characterized by a compact representation of a TU-game, derived by an additive pattern among coalitions:

- graph-restricted games (Myerson 1977, Owen 1986)
- component additive games (Curiel et al. 1993)
- MC-nets (Ieong and Shoham 2005)
- additive games with a permission structure (van den Brink et al. 2010)

Additive games

Sometimes, the worth of each coalition is computed from the values that single players can guarantee themselves by means of a mechanism describing the interactions of individuals within groups of players.

Simplest case: *additive games*

$$v(S) = \sum_{i \in S} v(i)$$

Example: Online purchase of n objects.

- compact way to represent interaction situations among players.
- may fail to reflect the importance of a subset of players in contributing to the value of the coalition they belong to.

Example: Online purchase of n objects: making a collective purchase might decrease the costs and the price that a coalition S should pay might therefore depend only on the price of a subset of purchased objects.

Generalized additivity

In some cases, the worth of a coalition $S \subseteq N$ is strongly related to the sum of the individual values over another subset $T \subseteq N$, not necessarily included in S.

Examples

- glove game
- airport games (Littlechild and Owen 1973; Littlechild and Thompson 1977)
- argumentation games (Bonzon et al. 2014)
- peer games (Branzei et al. 2002)
- mountain situations (Moretti et al. 2002)

glove game

- $N = L \cup R$, where the players in L that own a left-hand glove, and those in R with a right-hand glove.
- the worth of a coalition of players $S \subseteq N$ is defined as the number of pairs of gloves owned by the coalition S:

$$v(S) = min\{|S \cap L|, |S \cap R|\}$$

We can represent this game by assigning value 1 to each player and by describing the worth of each coalition S as the sum of single players' values over the smallest subset among $S \cap L$ and $S \cap R$.

In all the aformentioned models, the value of a coalition S of players is calculated as the sum of the single values of players in a subset of S.

On the other hand, in some cases the worth of a coalition might be affected by external influences and players outside the coalition might contribute, either in a positive or negative way, to the worth of the coalition itself.

Examples

- bankruptcy games (Aumann and Maschler 1985)
- maintanance problems (Koster 1999, Borm et al. 2001)

Generalized additive games (GAGs)

We introduce a general class of additive TU-games where the worth of a coalition $S \subseteq N$ is evaluated by means of an interaction filter, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in S.

- we show that well-known classes of TU-games can be represented in terms of GAGs
- we investigate the problem of computing the core, the nucleolus and the semivalues for specific families of GAGs.

Definition

We shall call *Generalized Additive Situation* (GAS) any triple $\langle N, v, \mathcal{M} \rangle$, where:

- N is the set of players;
- $v : N \rightarrow \mathbb{R}$;
- $\mathcal{M} : 2^N \rightarrow 2^N$, is the *coalitional map, which assigns a coalition* $\mathcal{M}(S)$ *to each coalition* $S \subseteq N$ *of players.*

Definition

Given the GAS $\langle N, v, \mathcal{M} \rangle$, the associated *Generalized Additive Game* (GAG) is the TU-game $(N, v^{\mathcal{M}})$ such that $v(\emptyset) = 0$ and for $S \neq \emptyset$:

$$v^{\mathcal{M}}(S) = \sum_{i \in \mathcal{M}(S)} v(i)$$

Examples (GAGs)

- Let w be a simple game. Then w is described by the GAG associated to $\langle N, v, \mathcal{M} \rangle$ with $v(i) = 1$ for all i and

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S & \text{if} \quad S \in W \\ \emptyset & \text{otherwise} \end{cases}$$

where W is the set of the winning coalitions in w.

- glove game.
- bankruptcy games.

An interesting subclass: basic GAGs

Let $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$, where $\mathcal{C}_i = \{F_i^1, \ldots, F_i^m, E_i\}$ is a collection of subsets of N such that $F_i^j \cap E_i = \emptyset$ for all $i \in N$ and for all $j = 1, \cdots, m$.

Definition

We denote by $\langle N, v, \mathcal{C} \rangle$ the *basic* GAS associated with the coalitional map $\mathcal{M}$ defined as:

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \ldots, S \cap F_i^m \neq \emptyset, S \cap E_i = \emptyset\}$$

and by $\langle N, v^{\mathcal{C}} \rangle$ the associated *basic* GAG.

- $F_i^1, \ldots, F_i^m$: sets of *friends* of player i;
- E_i: set of *enemies* of player i.

Decomposition of basic GAGs

The basic GAG $v^{\mathcal{C}}$ associated with a basic GAS can be decomposed in the following sense:

$$v^{\mathcal{C}} = \sum_{i=1}^{n} v^{\mathcal{C}_i},$$

where for $i = 1, \ldots, n$:

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if} \quad S \cap E_i = \emptyset, S \cap F_i^k \neq \emptyset, k = 1, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

Examples (basic GAGs)

- airport games (F_i, $E_i \neq \emptyset$)
- argumentation games ($F_i = \{i\}$, $E_i \neq \emptyset$; $F_i^1, \cdots F_i^m$, $E_i \neq \emptyset$)

- some classes of operation research games:
  - maintenance problems (F_i, $E_i = \emptyset$)
  - peer games ($F_i^1, \cdots F_i^m$, $E_i = \emptyset$)
  - mountain situations ($F_i = \{i\}$, $E_i \neq \emptyset$)

Maintanance problem

A mantainance problem is a couple (T, t), where:

- T=$(N \cup \{0\}, E)$ is a tree;
- 0 is the root of the tree having only one adjacent edge;
- $t : E \to \mathbb{R}^+$ is a nonnegative cost function on the edges of the tree.

Each vertex $i \in N$ is connected to the root 0 by a unique path P_i; we shall denote by e_i the edge in P_i that is incident to i.

A *precedence relation* $\preceq$ is defined by: $j \preceq i$ if and only if j is on the path P_i.

A *trunk* $R \subseteq N \cup \{0\}$ is a set of vertices which is closed under the relation $\preceq$, i.e. if $i \in R$ and $j \preceq i$, then $j \in R$. The cost of a trunk R is then defined as

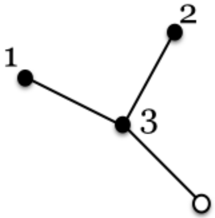
$$C(R) = \sum_{i \in R \setminus \{0\}} t(e_i).$$

Maintanance cost game

The associated *maintenance cost game* (N, c) is defined by

$$c(S) = \min\{C(R) : S \subseteq R \text{ and } R \text{ is a trunk}\}.$$

Example:



$P_1 = (1, 3, 0)$; $P_2 = (2, 3, 0)$; $P_3 = (3, 0)$.

$\{1, 3\}$ is a trunk; $\{1\}$ is not a trunk.

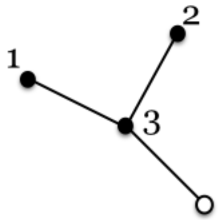
$c(\{1\}) = C(\{1, 3\}) = t(e_1) + t(e_3)$;
$c(\{1, 3\}) = C(\{1, 3\}) = t(e_1) + t(e_3)$;
$c(\{3\}) = C(\{3\}) = t(e_3)$;
$\vdots$

Let $F(i) = \{j \in N | i \preceq j\}$ be the set of *followers* of player i (note that $i \in F(i)$ for each $i \in N$).

Maintanance cost GAS

- $v(i) = t(e_i)$
- $\mathcal{C}_i = \{F_i, E_i\}$ such that $F_i = F(i)$ and $E_i = \emptyset$ for every $i \in N$

Example:



$F_1 = \{1\}$; $F_2 = \{2\}$; $F_3 = \{1, 2, 3\}$.

Some results

We provide some results on classical solution concepts for basic GAGs:

- some results on the core of $v^{\mathcal{C}}$ with empty set of enemies
- nucleolus of $v^{\mathcal{C}_i}$ with empty set of enemies
- formula for semivalues with multiple disjoint sets of friends $F_i^1, \cdots, F_i^m$, where $F_i^j \cap F_i^k = \emptyset$ for $j \neq k$
- concise formula for semivalues with one set of friends $F_i = \{i\}$
- concise formula for Shapley and Banzhaf values with two sets of friends: $F_i^1 = \{i\}$, F_i^2

Thank you for your attention