Power indices on cooperative games: semivalues.

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Cooperative game

N set of players.
$v : 2^N \to \mathbb{R}$ utility function, such that $v(\emptyset) = 0$.

Vector space given by all games on the finite set N: $\mathscr{G}_N \simeq \mathbb{R}^{2^n - 1}$

Simple games

- v is monotonic: $v(S) \leq v(T)$ if $S \subseteq T$
- $v : 2^N \to \{0,1\}$
- $v(N) = 1$

Unanimity games

For any coalition S

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Examples

Buyers and sellers (1)

A person wants to sell his car and evaluates it a value a. There are two potential buyers, that evaluates the car b and c, respectively. (suppose $b \leq c$ and to avoid trivial game $a < b$).

$$v(\emptyset) = 0 \qquad v(1) = a$$
$$v(2) = v(3) = v(\{2,3\}) = 0$$
$$v(\{1,2\}) = b \quad v(\{1,3\}) = v(N) = c$$

Buyers and sellers (2)

There are two people selling their own cars and only one potential buyer.

$$v(\emptyset) = v(1) = v(2) = v(3) = 0$$
$$v(\{1,3\}) = v(\{2,3\}) = v(N) = 1 \quad v(\{1,2\}) = 0$$

Bankruptcy Problem

$N = \{1, \dots, n\}$ is the set of players (i.e. of creditors), $c = (c_1, \dots, c_n)$ such that c_i is the money claimed by player i and E is the available capital; the bankruptcy condition is $E < \sum_{i \in N} c_i$.

$$v(S) = \max\{0, E - \sum_{i \in N \setminus S} c_i\} \text{ for every } S \subseteq N.$$

Talmud Bankruptcy Problem. A man, who was married to three wives, died and the *kethubah* (the money the groom should give to his bride) of the first wife was one-hundred *zuz*, the one of the second wife was two-hundred *zuz* while the third one deserved three-hundred *zuz*; but the estate was three-hundred *zuz*.

$$\begin{array}{ccc} v(\emptyset) = 0 & & v(N) = 300 \\ v(1) = 0 & v(2) = 0 & v(3) = 0 \\ v(\{1,2\}) = 0 & v(\{1,3\}) = 100 & v(\{2,3\}) = 200 \end{array}.$$

Examples

Weighted Majority Game

Given a positive number q and non negative integers $w_1, \ldots, w_n$ the game $v = [q; w_1, w_2, \ldots, w_n]$ is defined by

$$v(T) = \begin{cases} 1 & \text{if } w(T) = \sum_{i \in T} w_i \geq q \\ 0 & \text{otherwise} \end{cases}.$$

UN Security Council 5 permanent members and 10 non permanent members. A motion is accepted if it gets at least 9 votes, including all the votes of the permanent members.

$$v = [39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

Solution

Solution

$\varphi : \mathscr{G}_N \to \mathbb{R}^n$ s.t. $\varphi_i(v)$ is the amount given to player i in game v.

Properties:

- *linearity:* $\varphi(v+w) = \varphi(v) + \varphi(w)$ *e* $\varphi(\lambda v) = \lambda \varphi(v)$;
- *positivity:* if v is monotonic then $\varphi_i(v) \geq 0 \forall i$;
- *efficiency:* $\sum_{i \in N} \varphi_i(v) = v(N)$;
- *symmetry:* for any permutation ϑ on N

  $$\varphi_i(\vartheta v) = \varphi_{\vartheta(i)}(v)$$

  where $(\vartheta v)(S) = v(\vartheta(S))$;
- *dummy player:* $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all S, then $\varphi_i(v) = v(\{i\})$.

Solution concepts

Imputation

φ that are efficient and individually rational, i.e.

$$I(v) = \sum_{i \in N} \varphi_i(v) = v(N) \text{ and } \varphi_i(v) \geq v(\{i\})$$

for all $i \in N$.

Core

φ that are efficient and coalitionally rational, i.e.

$$C(v) = \{\varphi : \sum_{i \in N} \varphi_i(v) = v(N) \text{ and } \sum_{i \in S} \varphi_i(v) \geq v(S) \text{ for all } S \subseteq N\}$$

Power indices

Shapley

$$\sigma_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

It satisfies linearity, symmetry, dummy player and efficiency.

Banzhaf

$$\beta_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{2^{n-1}} [v(S \cup \{i\}) - v(S)]$$

It satisfies linearity, symmetry and the dummy player property.

Examples

Talmud Bankruptcy Problem

$$\begin{array}{ccc} v(\emptyset) = 0 & v(i) = 0 & v(N) = 300 \\ v(\{1,2\}) = 0 & v(\{1,3\}) = 100 & v(\{2,3\}) = 200 \end{array}.$$

The Shapley value is

$$\sigma_1 = 50 \qquad \sigma_2 = 100 \qquad \sigma_3 = 150$$

UN Security Council

$$v = [39; 7,7,7,7,7,1,1,1,1,1,1,1,1,1,1].$$

If i is a permanent member and j is a non permanent member

$$\sigma_i(v) = \frac{421}{2145} \qquad \sigma_j(v) = \frac{4}{2145}$$

$$\beta_i(v) = \frac{53}{1024} \qquad \beta_j(v) = \frac{21}{4096}$$

Senato della Repubblica

$$v = [161; 108, 91, 50, 20, 16, 16, 10, 10]$$

Party	% Seats	Shapley	Banzhaf
Partito Democratico	33.65	33.8	32.14
Popolo delle Libertà	28.35	26.19	25.00
5 Stelle	15.58	21.42	23.21
Scelta Civica	6.23	5.24	5.36
Lega Nord	4.98	3.33	3.57
Misto	4.98	3.33	3.57
Grandi Autonomie	3.11	3.33	3.57
Per le Autonomie	3.11	3.33	3.57

Probabilistic value

Proabilistic value

$\varphi : \mathscr{G}_N \to \mathbb{R}^n$ that satisfies the following properties: linearity, positivity and dummy player.

Theorem

*φ is a **probabilistic value** iff for all $i \in N$ there are $\{p_S^i\}_{S \subseteq N \setminus \{i\}}$ such that $p_S^i \geq 0$, $\sum_{S \subseteq N \setminus \{i\}} p_S^i = 1$*

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} p_S^i [v(S \cup \{i\}) - v(S)].$$

Probabilistic + efficiency $\implies$ quasivalue.
Probabilistic + symmetry $\implies$ semivalue.

Semivalue

$\varphi : \mathscr{G}_N \to \mathbb{R}^n$ linear, positive, symmetry and dummy player.

Theorem

*φ is a **semivalue** iff there are $\{p_s\}_{s=0,\dots n-1}$ such that $p_s \geq 0$, $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$*

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} p_s [v(S \cup \{i\}) - v(S)].$$

Other semivalues

q-binomial

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} q^s (1-q)^{n-s-1} [v(S \cup \{i\}) - v(S)]$$

Regular semivalues

$p_s > 0$ for all $s = 0, \dots, n-1$.

Dictatorial

$$p_0 = 1 \qquad p_s = 0$$

for all $s = 1, \dots, n-1$.

Marginal

$$p_s = 0 \qquad p_{n-1} = 1$$

for all $s = 0, \dots, n-2$.

Other semivalues

Modified Values

If φ is a semivalue with coefficients p_s, the semivalue $\varphi_{m_1}^{m_2}$ is defined by the coefficients

$$p'_s = \begin{cases} \frac{p_s}{\sum_{j=m_1}^{m_2} p_j \binom{n-1}{j}} & \text{if } s \in [m_1, m_2] \\ 0 & \text{otherwise.} \end{cases}$$

Modified Shapley:

$$p'_s = \frac{1}{(m_2 - m_1 + 1)\binom{n-1}{s}}$$

Modified Banzhaf:

$$p'_s = \frac{1}{\sum_{j=m_1}^{m_2} \binom{n-1}{j}}$$

Senato della Repubblica

	PD	PdL	5 Stelle	Sc. Civica	Altri
$\beta = \beta_0^7$	32.14	25	23.21	5.357	3.571
β_0^1, β_1^1	50	50	0	0	0
β_0^2, β_1^2	37.5	25	18.75	6.25	3.125
$\beta_3^3, \beta_3^4, \beta_4^4$	30.0	25.0	25.0	5.0	3.75
β_2^4, β_3^5	31.05	24.74	24.21	5.263	3.684
β_5^6, β_5^7	37.5	25	18.75	6.25	3.125
β_6^6, β_6^7	50	50	0	0	0

Table: Modified Banzhaf index

Microarray game

Let $M = (m_{ij})$ be a matrix such that $m_{ij} \in \{0, 1\}$. For every j define $S_j = \{i : m_{ij} = 1\}$ and v^j

$$v^j(T) = \begin{cases} 1 & \text{if } S_j \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

The *microarray game* associated to M is defined as

$$v = \frac{1}{m} \sum_{j=1}^{m} v^j.$$

The columns represent the patients while the rows represent the genes, that are the players of this game.
$m_{ij} = 1 \implies$ the gene i is abnormally expressed in patient j,
$m_{ij} = 0 \implies$ the gene i is normally expressed in patient j

Example

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$v(\{1\}) = v(\{2\}) = \frac{1}{4} \qquad v(\{3\}) = 0$$

$$v(\{1,2\}) = \frac{3}{4} \qquad v(\{1,3\}) = \frac{1}{4}$$

$$v(\{2,3\}) = \frac{2}{4} \qquad v(N) = 1$$

Shapley and Banzhaf values give different rankings of genes, because they have a different behaviour on unanimity games.

$$\sigma_i(u_S) = \frac{1}{s} \qquad \beta_i(u_S) = \frac{1}{2^{s-1}}$$

a-value

$$\sigma_i^a(u_S) = \begin{cases} \frac{1}{s^a} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

where u_S is the simple game such that $u_S(T) = 1$ iff $S \subseteq T$.

$\mathscr{G}$ space of all finite games

U universe of players.
$v : 2^U \to \mathbb{R}$ and there is a finite$N$ such that $\forall S \subset U,\ v(S) = v(S \cap N)$.

$\mathscr{AG}$ - Additive games: $v(S \cup T) = v(S) + v(T)$

Semivalues on $\mathscr{G}$

$\varphi : \mathscr{G} \to \mathscr{AG}$ that satisfies:

- linearity;
- symmetry: $\varphi \circ \vartheta = \vartheta \circ \varphi$ for all permutation ϑ;
- monotonicity: v monotonic $\implies \varphi(v)$ monotonic;
- axis projection: if $v \in \mathscr{AG}$ then $\varphi(v) = v$.

Theorem

φ is a semivalue on $\mathscr{G}$ iff there is a probability distributiion ξ over $(0,1)$ such that

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \left(\int_0^1 t^s (1-t)^{n-s-1} d\xi(t) \right) [v(S \cup \{i\}) - v(S)]$$

for each game v with support N.

Unanimity Game

For any coalition S:

$$u_S(T) = \begin{cases} 1 & S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Generating Semivalues

Theorem

The function φ defined on the basis of unanimity games

$$\varphi_i^\alpha(u_S) = \begin{cases} \alpha_s & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity is a semivalue on $\mathscr{G}$ iff $\alpha_1 = 1$ and the sequence α_s is completely monotonic.

A sequence $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic if $\mu_n \geq 0$ and

$$(-1)^k \Delta^k \mu_n = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu_{n+k-j} \geq 0$$

for all $k, n = 0, 1, 2, \ldots$.

Proof

$$p_s^n = \sum_{k=0}^{n-s-1} \binom{n-s-1}{k} (-1)^k \alpha_{s+k+1}$$

So we get

$$\sum_{s=0}^{n-1} \binom{n-1}{s} p_s^n = 1 \iff \alpha_1 = 1$$

$$p_s^n \geq 0 \iff (-1)^k \Delta^k \alpha_t \geq 0$$

a-values

$$\sigma_i^a(u_S) = \begin{cases} \frac{1}{s^a} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Shapley value if $a = 1$.

q-binomial values

$$\phi_i(u_S) = \begin{cases} q^{s-1} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Banzhaf value if $q = \frac{1}{2}$.

Not regular semivalues

Proposition

Let $\{\mu_k\}_{k=0}^{+\infty}$ be a completely monotonic sequence. Then

$$(-1)^k \Delta^k \mu_n > 0$$

for every n, k unless the sequence is constant except at most for the first term.

A semivalue φ on $\mathscr{G}$ is regular iff $\varphi|_N$ is a regular semivalue on $\mathscr{G}_N$ for all N. A semivalue that is not regular is irregular.

Irregular semivalues

The irregular semivalues are generated by $\alpha = (1, q, \dots q)$ for any $q \in [0,1]$ and their weighting coefficients are $p^n = (1-q, 0, \dots, 0, q)$ for all n.
If $q = 1$ we find the **marginal value**.
If $q = 0$ we find the **dictatorial value**.

A function $f(x)$ is *completely monotonic* if $(-1)^n f^{(n)}(x) \geq 0$, $\forall n \geq 0$.

We can use completely monotonic functions to define completely monotonic sequences:

Theorem

Let $f(x)$ be a completely monotonic function in $[a, +\infty)$ and let δ be any fixed positive number, then

$$\mu_n = \{f(a + n\delta)\}_{n=0}^{+\infty}$$

is a completely monotonic sequence.

$$f(x) = e^{\frac{a}{x}} \qquad f(x) = ln(b + \frac{c}{x}) \qquad f(x) = \frac{1}{(d + ex)^{\gamma}}$$

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