Power indices: a measure of centrality in networks

Giulia Cesari

Dipartimento di Matematica, Politecnico di Milano
LAMSADE, Université Paris Dauphine

Milano, 10-12-2014

Cooperative TU-games

Coalitional game

A pair (N, v):

- N is the set of players;
- $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$.

- **Simple games**: $v : 2^N \to \{0, 1\}$
- **Convex games**: $v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \; \forall S, T \subseteq N$
- **Super-additive games**: $v(S \cup T) \geq v(S) + v(T) \; \forall S, T\colon \; S \cap T = \emptyset$

The Shapley value

How to convert information about the worth that subsets of the player set can achieve, into a personal attribution (of payoff) to each of the players?

Shapley (1953)

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right), \ \forall i \in N$$

- $s = |S|$ is the cardinality of coalition S.
- $n = |N|$ is the cardinality of the players set N.

Regular semivalues

Probabilistic indices

$$\Psi_i(v) = \sum_{S \subseteq N \setminus \{i\}} p_i^n(S)\big(v(S \cup \{i\}) - v(S)\big) \;\; \forall i \in N,$$

- *semivalues*: $p_i^n(S)$ does not depend on the player i and depends only on the size of the coalition S: $p_i^n(S) = p^n(s) := p_s^n$.
- *regular semivalue*: $p_s^n > 0$ for all s.

Semivalues as power indices

- Shapley (1953): axiomatic characterization on TU-games.
- Shapley and Shubik (1954): *power index* on simple games.

Power indices

- measure of the influence of a player on the outcome of the game
- a priori evaluation of the probability for a player to play a relevant role in the game.

Cooperative games with restricted communications

Coalitional games

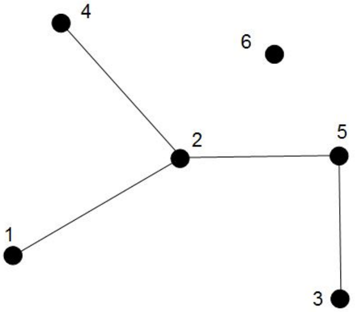
- all coalitions are feasible

However, this is not the case in many real world situations:

- social interaction in a group of people
- political alliances within parties
- economic exchanges among firms
- research collaborations
- genes interactions in a cell

How can we model these restrictions on the feasible coalitions ?

- by introducing a *network* structure between the players



Network

- nodes: players in the game
- edges: direct interaction between nodes

Communication situations

Typical situation: restriction of communication possibilities between players.

A **communication situation** or **graph game** is a triple (N, v, Γ):

- (N, v) is a coalitional game;
- $\Gamma = (N, E)$ is an undirected graph with N as set of vertices.

- an indirect communication between i and j is possible if there is a path that connects them;
- if $\{i, j\} \in E$, then i and j can communicate directly.

Solution concepts for communication situations

How do the communication constraints influence the allocation rules?

First approach

The communication constraints determine how a coalition is *evaluated*.

- Myerson value (1977); Position value (1982);

Second approach

The communication constraints determine how the coalitions are to be *formed*.

- Demange (2004); Herings et al. (2008); Béal et al. (2012);

Second approach: the average tree solution

Shapley value

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \big(v(S \cup \{i\}) - v(S)\big), \ \forall i \in N$$

Basic idea

Orderings of the players that induce unconnected coalitions are ruled out.

$\Rightarrow$ *rooted spanning trees*

First approach: the Myerson value

Graph-restricted game

Myerson (1977) introduced the *graph-restricted game* v^Γ, defined by:

$$v^\Gamma(S) = \sum_{T \in C_{\Gamma_S}} v(T),$$

where C_{Γ_S} is the set of connected components in Γ_S.

The **Myerson value** $\mu(v)$ is the Shapley value of the graph-restricted game:

$$\mu(N, v, \Gamma) = \Phi(v^\Gamma)$$

The Myerson value: characterization

Property 1 (Component efficiency)

For every $C \in C_\Gamma$, $\sum_{i \in C} \mu_i(N, v, \Gamma) = v(C)$.

Property 2 (Fairness)

For every $i, j \in E$:

$$\mu_i(N, v, \Gamma) - \mu_i(N, v, \Gamma \backslash \{i, j\}) = \mu_j(N, v, \Gamma) - \mu_j(N, v, \Gamma \backslash \{i, j\})$$

N.B. Fairness is also referred to as *balanced contributions* property.

A simple example

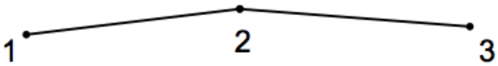
Weighted majority game: $(\{1,2,3\}, v)$ with quota $q = 2/3$.
Players' weights: $w_1 = w_3 = 40\%$, $w_2 = 20\%$.

$v(1,3) = v(1,2,3) = 1$ and $v(S) = 0$ for all other coalitions.

↓

$$\phi(v) = (1/2, 0, 1/2)$$

A simple example (2)



$v^{\Gamma}(1,2,3) = 1$ and $v^{\Gamma}(S) = 0$ for all other coalitions.

$$\mu(N, v, \Gamma) = \phi(v^{\Gamma}) = (1/3, 1/3, 1/3).$$

The position value

A different approach is found in Meessen (1988) and Borm et al. (1992).

Link game

For $A \subseteq E$:

$$v^L(A) = \sum_{T \in C_{\Gamma_A}} v(T),$$

where C_{Γ_A} is the set of connected components in Γ_A.

The **position value** $\pi(N, v, \Gamma)$ is defined as:

$$\pi_i(N, v, \Gamma) = \frac{1}{2} \sum_{a \in A_i} \Phi_a(v^L) \quad \forall i \in N,$$

where A_i is the set of all links for which player i is an endpoint.

The position value: characterization

Property 1 (Component efficiency)

For every $T \in C_\Gamma$:

$$\sum_{i \in T} \pi_i(N, v, \Gamma) = v(T).$$

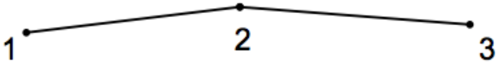
Property 2 (Balanced total threats)

For every $i, j \in N$:

$$\sum_{a \in A_j} \pi_i(N, v, \Gamma) - \pi_i(N, v, \Gamma \setminus \{a\}) = \sum_{a \in A_i} \pi_j(N, v, \Gamma) - \pi_j(N, v, \Gamma \setminus \{a\})$$

A simple example (3)

Weighted majority game: $(\{1,2,3\}, v)$ with quota $q = 2/3$.
Players' weights: $w_1 = w_3 = 40\%$, $w_2 = 20\%$.



- Shapley value: $\phi(v) = (1/2, 0, 1/2)$
- Myerson value: $\mu(N, v, \Gamma) = (1/3, 1/3, 1/3)$

$v^L(\{1,2\}) = v^L(\{2,3\}) = 0$ and $v^L(\{1,2\},\{2,3\}) = 1$.

$$\pi(v) = (1/4, 1/2, 1/4)$$

Myerson value and position value as *power indices*:

- measure of the influence of a player in a graph game

Application to *centrality analysis*: centrality as variation of power in a game due to the network structure.

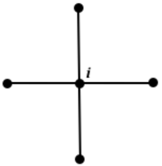
What does *centrality* mean?

Centrality in a network: classical measures

Centrality - Bavelas (1948): studies on communication within groups

Basic idea: the *hub* of a star is the most central position.

- i can communicate directly with many other nodes.
- a lot of nodes need i to be able to communicate between each other.
- i is close to many other nodes.

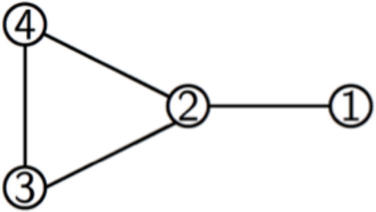


⇒ *degree centrality*

⇒ *betweenness centrality*

⇒ *closeness centrality*

Centrality measures: an example

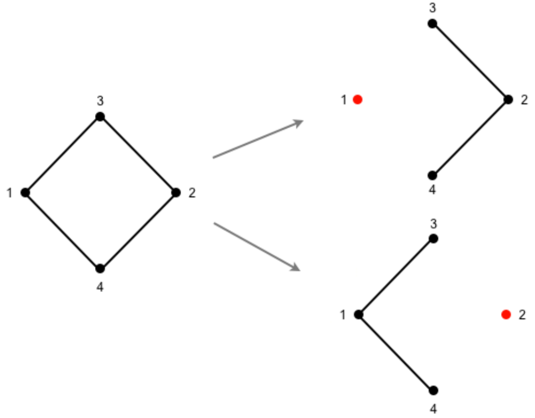


Nodes	1	2	3	4
degree	1	3	2	2
betweenness	0	2	0	0
closeness	$\frac{3}{5}$	1	$\frac{3}{4}$	$\frac{3}{4}$

Classical measures: limitations

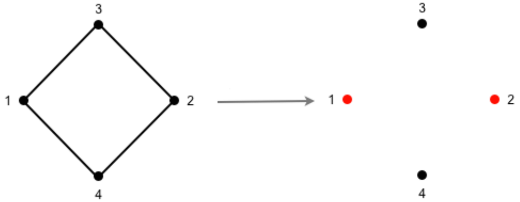
Classical centrality measures may fail to reflect the role of subsets of nodes.

Example: communication network (failure of a node)



Classical measures: limitations

Example: communication network (failure of a couple of nodes)



Game theoretic network centrality

An approach to centrality analysis using game theory has been proposed as theoretical framework to face such limitations.

Game theoretic network centrality

Gómez et al. (2003)

- *Coalitional game* (N, v) $\Rightarrow$ Shapley value of v.
- *Graph-restricted game* (N, v^Γ) $\Rightarrow$ Shapley value of v^Γ.

Centrality measure γ

$$\gamma_i(v, \Gamma) = \phi_i(v^\Gamma) - \phi_i(v) \quad \text{for all } i \in N$$

Centrality as the variation of power due to the restriction of communication imposed by the network.

Properties of γ

If v is symmetric and convex:

- Fairness: removing an edge changes the centrality of both incident nodes by an equal amount.
- Stability: removing an edge decreases the centrality of both incident nodes.
- Independence of the remaining connected components. (CFR. *degree*)
- Isolated nodes have minimal centrality. (CFR. *degree*)
- Of all graphs with n nodes the max centrality is attained by the hub of a star. (CFR. *degree, betweenness, closeness*)
- Of all connected graphs with n nodes, the min centrality is attained by the end nodes in a chain. (CFR. *degree, betweenness*)
- In a chain, centrality increases from the end nodes to the median node. (CFR. *degree, betweenness, closeness*)

Top-k nodes problem

Suri and Narahari (2008)

- Context: co-autorship network, diffusion of information, viral marketing etc.
- Top k-nodes problem is NP-hard.
- The game: the value of a coalition is the number of neighbors
- The top-k nodes: the nodes with the highest Shapley value
- Efficient approximate algorithm tested on a co-autorship data set (8361 authors)

Top-k nodes problem

Aadithya et al. (2010)

- Exact formula for the Shapley value of the game introduced by Suri and Narahari (2008):

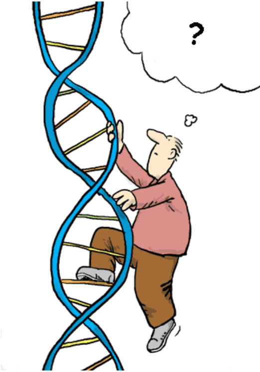
$$\phi_i(N, v, \Gamma) = \sum_{j \in fringe(i)} \frac{1}{1 + deg(j)}$$

Centrality in a biological network

Problem

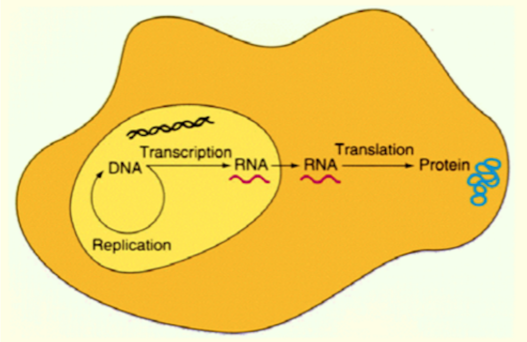
Identification of genes that play a relevant role in a certain biological process.

- Moretti et al. (2007): *microarray games*: a relevance index for genes
- Lucchetti et al. (2010): Shapley and Banzhaf in microarray games
- Moretti et al. (2010): centrality of genes in a biological network



Biology "in pillole"

- Every organisms is made of *cells*
- The *proteins* are the building blocks of the cells and act as enzymes in the biochemical reactions
- The *genes* are responsible for the synthesis of proteins



- The interactions between genes, RNA and proteins are described by *gene regulatory network* o *gene regulatory pathway*

Relevance of genes in a biological network

Coexpression network

- the nodes represent genes
- connection between nodes is determined by coexpression (correlation of genes' profiles)

Centrality in a coexpression network

Centrality analysis is an important tool for the interpretation of genes interaction.

- Jeong et al. (2001): "Lethality and centrality in protein networks."
- Carlson et al. (2006): "Gene connectivity, function, and sequence conservation: predictions from modular yeast co-expression networks."

Game theoretic network centrality

Problem

Identification of genes that play a relevant role in a certain biological process.

Moretti et al. (2010)

- *Association game*: (N, v), $\forall S \subseteq N$ $v(S)$ is the number of *key genes* that interact only with genes in S.
- *Graph-restricted game*: (N, v^Γ) introduced by Myerson.

Centrality measure γ

$$\gamma_i(N, v, \Gamma) = \phi_i(v^\Gamma) - \phi_i(v) \quad \forall i \in N$$

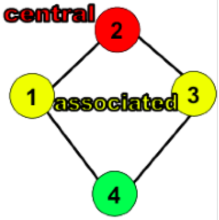
An example

Association game

1	LOSE
2	LOSE
3	LOSE
1 2	LOSE
2 3	LOSE
1 3	WIN
1 2 3	WIN

Co-expression network game

1	LOSE
2	LOSE
3	LOSE
1 2	LOSE
2 3	LOSE
1 3	LOSE
1 2 3	WIN



	Shapley value		
	Asso	Co-expr.	Diff.
1	1/2	1/3	-1/6
2	0	1/3	1/3
3	1/2	1/3	-1/6

References

Aadithya KV, Ravindran B, Michalak TP, Jennings NR (2010) *Efficient computation of the Shapley Value for centrality in networks* WINE 2010, LNCS, 6484, 1-13.

Bavelas A (1948) *A mathematical model for group structures.* Human Organization, 7, 16-30.

Beal, S, Remila E, Solal P (2012) *Compensations in the Shapley value and the compensation solutions for graph games.* International Journal of Game Theory 41.1: 157-178.

Bavelas A A mathematical model for group structures. (1948) Human organization 7.3: 16-30.

Borm P, Owen G, Tijs S (1992) *On the position value for communication situations*. SIAM Journal on Discrete Mathematics. 5(3):305-320.

Carlson M RJ, Zhang B, Fang Z, Mischel PS, Horvath S, Nelson SF (2006) *Gene connectivity, function, and sequence con- servation: predictions from modular yeast co-expression networks.* BMC Genomics, 7, 40.

Demange G (2004) *On group stability in hierarchies and networks.* Journal of Political Economy 112.4: 754-778.

Freeman LC (1979) *Centrality in social networks: conceptual clarification.* Social Networks, 1, 215-239.

Gomez D, Gonzalez-Aranguena E, Manuel C, Owen G, Del Pozo M, Tejada J: (2003) *Centrality and power in social networks: a game theoretic approach.* Mathematical Social Sciences, 46, 27-54.

Hamiache G (1999) *A value with incomplete communication. Games and Economic Behavior*. 26(1):59-78.

Herings P, van der Laan G, Talman D (2008) *The average tree solution for cycle-free graph games.* Games and Economic Behavior 62.1: 77-92.

Herings P, van der Laan G, Talman AJJ, Yang Z (2010) *The average tree solution for cooperative games with communication structure.* Games and Economic Behavior 68.2: 626-633.

References

Lucchetti R, Moretti S, Patrone F, Radrizzani P (2010) *The Shapley and Banzhaf values in microarray games* Computers & Operations Research 37.8:1406-1412.

Jeong H, Mason SP, Barabasi AL (2001) *Lethality and centrality in protein networks.* Nature, 411, 41-42.

Meesen R (1988) *Communication games*. Master's Thesis (in Dutch), Department of Mathematics, University of Nijmegen.

Moretti S, Patrone F, Bonassi S (2007) *The class of microarray games and the relevance index for genes.* Top, 15, 256-280.

Moretti S, Fragnelli V, Patrone F, Bonassi S (2010) *Using coalitional games on biological networks to measure centrality and power of genes.* Bioinformatics, 26, 2721-2730.

Myerson RB (1977) *Graphs and cooperation in games*. Mathematics of operations research. 2(3):225-229.

Shapley (1952) *A value for n-person games*. Technical report, DTIC Document.

Shapley LS, Shubik M (1954) *A method for evaluating the distribution of power in a committee system.* American Political Science Review 48.03: 787-792.

Slikker M (2000) *Decision making and cooperation restrictions*. Technical report,Tilburg University.

Slikker M (2005) *A characterization of the position value*. International Journal of Game Theory. 33(4):505-514.

Suri NR, Narahari Y (2008) *Determining the top-k nodes in social networks using the Shapley Value.* AAMAS 2008, 1509-1512.