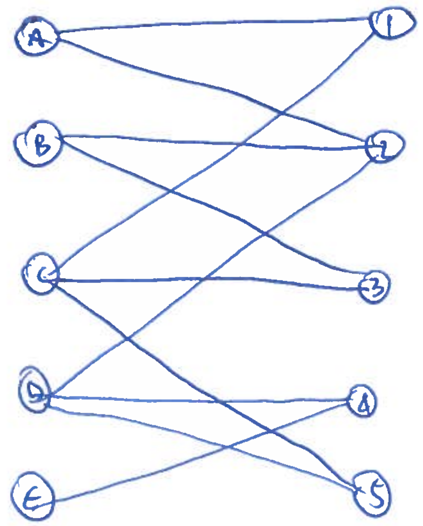
MATCHING MARKETS

• MATCHING

DEFINITION: BIPARTITE GRAPH: THE VERTICES ARE DIVIDED IN TWO GROUPS SUCH THAT NO EDGE BETWEEN TWO VERTICES OF THE SAME GROUP IS PRESENT

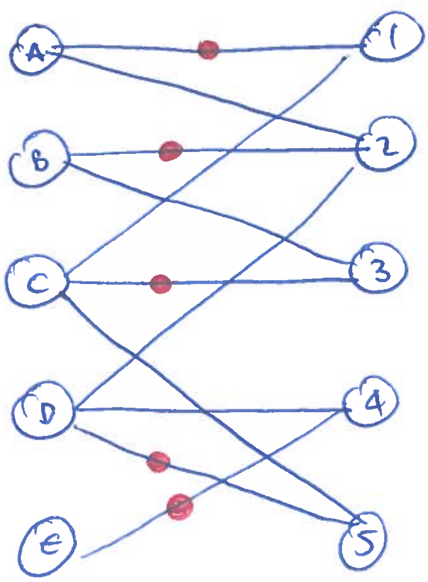
EXAMPLE:



DEFINITION: A MATCH IS A COLLECTION OF EDGE SUCH THAT FOR EACH VERTEX THERE IS AT MOST ONE EDGE

DEFINITION: PERFECT MATCH OF A BIPARTITE GRAPH IN WHICH THE TWO GROUPS HAVE THE SAME NUMBER OF ELEMENTS IT IS A MATCH IN WHICH THERE IS ONE EDGE PER VERTICES

EXAMPLE:



PERFECT MATCH

FORMULATION: A MATCHING PROBLEM CAN BE WRITTEN AS

$$x_{iT} \in \{0,1\} \quad \forall i \in G_1, \forall T \in G_2$$

$$\sum_{i \in G_1} x_{iT} = 1 \quad \forall T \in G_2$$

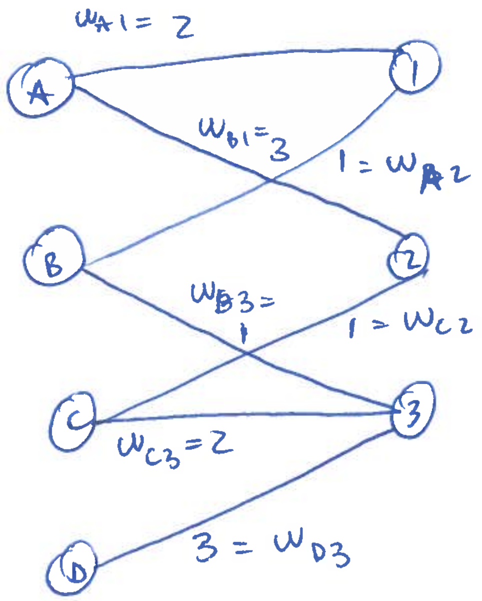
$$\sum_{T \in G_2} x_{iT} = 1 \quad \forall i \in G_1$$

$x_{iT} \le 0$ IF NO EDGE CONNECTS i AND T $\forall i \in G_1, \forall T \in G_2$

OBSERVATION: THE FORMULATION IS AN INTEGER LINEAR PROGRAM (ILP). ILP IS GENERICALLY NP-COMPLETE.

DEFINITION: THE ASSIGNMENT PROBLEM EXTENDS THE MATCHING PROBLEM ALLOWING WEIGHTS OVER THE EDGES AND NOT REQUIRING PERFECT MATCHES

EXAMPLE:



A PERFECT MATCH IS NOT POSSIBLE GIVEN THAT THE GROUPS HAVE A DIFFERENT NUMBER OF ELEMENTS

FORMULATION: THE SOLUTION OF AN ASSIGNMENT PROBLEM CAN BE OBTAINED BY SOLVING

$$\max \sum_{i \in G_1} \sum_{T \in G_2} w_{iT} x_{iT}$$

$$\sum_{i \in G_1} x_{iT} \le 1 \quad \forall T \in G_2$$

$$\sum_{T \in G_2} x_{iT} \le 1 \quad \forall i \in G_1$$

$x_{iT} = 0$ IF NO EDGE BETWEEN i AND T

$$x_{iT} \in \{0,1\}$$

PROPERTY: THE ASSIGNMENT PROBLEM CAN BE SOLVED IN POLYNOMIAL TIME BY THE HUNGARIAN ALGORITHM.

PROPERTY: GIVEN THAT THE MATCHING PROBLEM IS A PARTICULAR CASE OF THE ASSIGNMENT PROBLEM, ALSO THE MATCHING PROBLEM CAN BE SOLVED IN POLYNOMIAL TIME.

ALGORITHM: THE HUNGARIAN ALGORITHM WORKS AS FOLLOWS:
THE INPUT OF THE PROBLEM IS A SQUARE MATRIX IN WHICH THE WEIGHTS/COSTS ARE REPORTED

	Napoli	Florence	Bari	Palermo
COMO	150	100	300	400
MILAN	100	50	300	250
TURIN	200	100	400	400
ROME	50	50	100	150

(COST TO MINIMIZE)

OBSERVATION: IF WE ADD OR SUBTRACT A CONSTANT FROM ALL THE ENTRIES OF A ROW OR OF A COLUMN OR THE NEW PROBLEM IS THE OPTIMAL SOLUTION OF THE NEW PROBLEM IS ALSO AN OPTIMAL SOLUTION OF THE ORIGINAL PROBLEM

THE ALGORITHM IS IN FIVE STEPS

STEP 1: FOR EACH ROW, FIND THE SMALLEST ENTRY AND SUBTRACT IT FROM ALL THE ENTRIES

STEP 2: DO THE SAME FOR EACH COLUMN

STEP 3: FIND THE MINIMAL COVER OF ROWS OR COLUMNS OVER THE ZERO ELEMENTS OF THE MATRIX

STEP 4: IF THE NUMBER OF ROWS AND COLUMNS IS SMALLER THAN THE SIZE OF THE MATRIX, THEN IDENTIFY THE SMALLEST ENTRY NOT COVERED, SUBTRACT IT FROM ALL THE UNCOVERED ROWS, AND ADD IT TO ALL COVERED COLUMNS AND GO TO STEP 3

EXAMPLE:

150	100	300	400
100	50	300	250
200	100	400	400
50	50	100	150

STEP 1:

50	0	200	300
50	0	250	200
100	0	300	300
0	0	50	100

STEP 2:

50	0	150	200
50	0	200	100
100	0	250	200
0	0	0	0

STEP 3:

50	0	150	200
50	0	200	100
100	0	250	200
0	0	0	0

2 COVERING ROWS/COLUMNS

STEP 4:

THE MINIMUM UNCOVERED ENTRY IS 50

0	0	100	150
0	0	150	50
50	0	200	150
0	50	0	0

STEP 3:

0	0	100	150
0	0	150	50
50	0	200	150
0	50	0	0

3 COVERING ROWS/COLUMNS

STEP 4: THE MINIMUM UNCOVERED ENTRY IS 50

0	0	50	100
0	0	100	0
50	0	150	100
50	100	0	0

THE OPTIMAL ASSIGNMENT IS

COMO — NAPLE

TURIN — FLORENCE

ROME — BARI

MILAN — PALERMO

MAXIMIZATION: IF WE WANT TO MAXIMIZE, IT IS NECESSARY TO CHANGE THE SIGN TO THE ENTRIES

-150	-100	-300	-400
-100	-50	-300	-250
-200	-100	-400	-400
-50	-50	-100	-150

COMPLEXITY: THE COMPLEXITY IS $O(n^4)$ WHERE n IS THE SIZE OF THE MATRIX

• Prices and market-clearing property

Scenario:

- Consider a matching problem in which there are buyers that must be matched with sellers
- Each seller J sells a good for a price $p_J \geq 0$
- Each buyer i has a valuation v_{iJ} over the good of seller J
- The utility of a buyer from a match is $v_{iJ} - p_J$
- If $v_{iJ} - p_J < 0$ the match is not possible

Example:

Prices	Sellers	Buyers	Valuations (a, b, c)
5	a	x	12, 4, 2
2	b	y	8, 7, 6
0	c	z	7, 5, 2

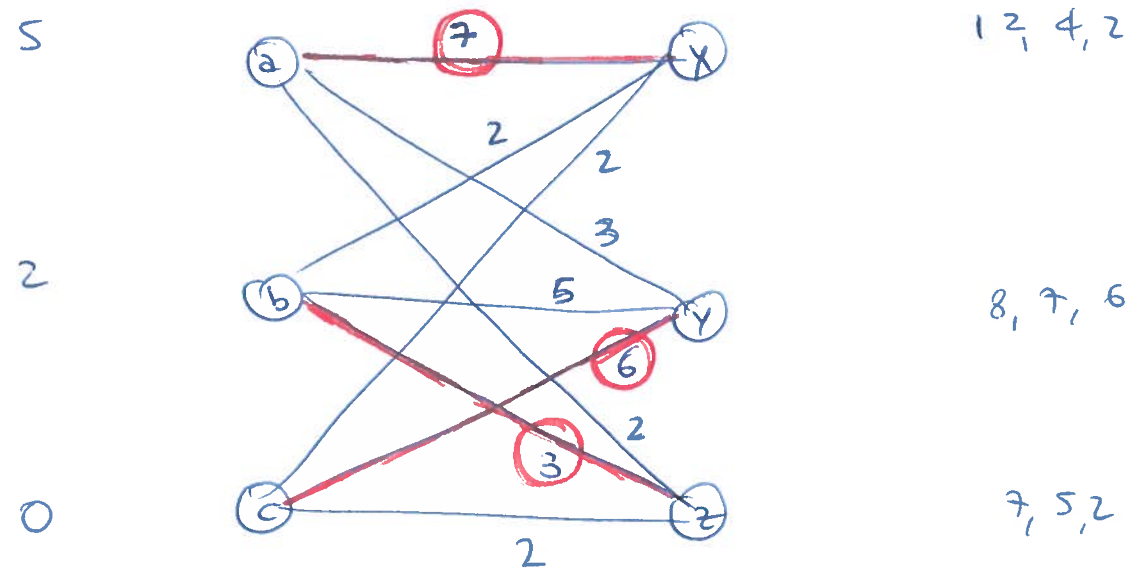
All the links are possible $(v_{iJ} - p_J \geq 0 \quad \forall i, J)$

Definition: Preferred-seller graph on buyers and sellers is the set of edges connecting each buyer and her preferred buyer(s).

Definition: A set of prices p_J is market-clearing if the resulting preferred-seller graph has a perfect matching.

EXAMPLE

PRICES SELLERS BUYER VALUATIONS (a, b, c)



THE PREFERRED GRAPH IS REPORTED ABOVE

IT ADMITS A PERFECT MATCHING ⟹ MARKET-CLEARING

PROPERTY: MARKET-CLEARING PRICES ALWAYS EXIST

OBSERVATION: A SIMPLE WAY TO FIND MARKET-CLEARING PRICES IS TO FIND THE OPTIMAL ALLOCATION IN TERMS OF VALUATIONS AND APPLY THE VALUES OF THE MATCH

EXAMPLE: OPTIMAL ASSIGNMENT

$(a, x) \longrightarrow 12$

$(b, z) \longrightarrow 5$

$(c, y) \longrightarrow 6$

PRICES

$P_a = 12$

$P_b = 5$

$P_c = 6$

OBSERVATION:
- THE PRICES COMPUTED WITH THE OPTIMAL ASSIGNMENT ARE THE LARGEST VALUES
- WITH THOSE VALUES THE BUYERS HAVE A UTILITY OF ZERO
- THERE CAN BE MARKET-CLEARING PRICES DIFFERENT FROM THE ABOVE ONES

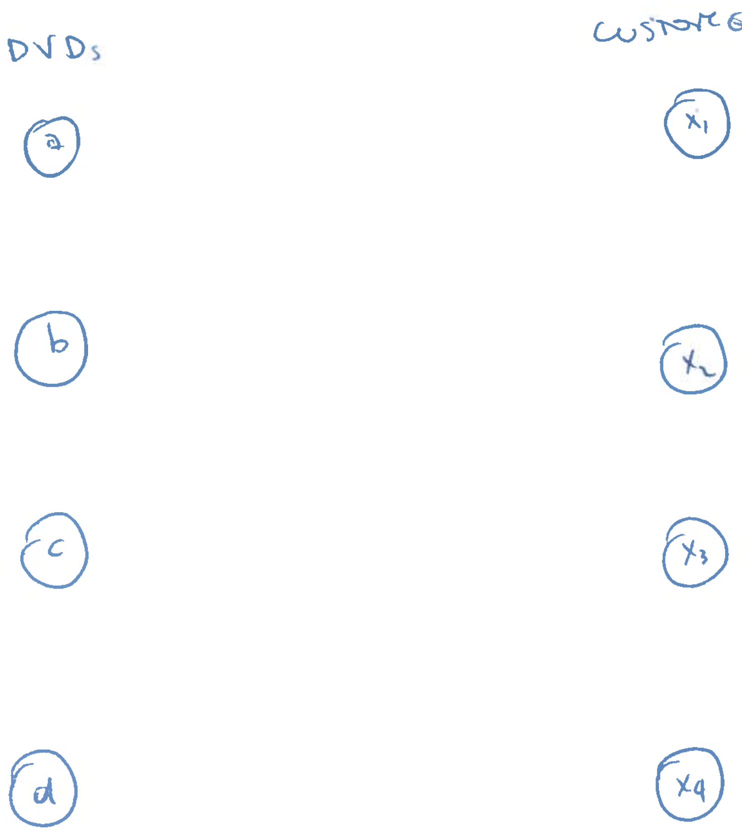
ALGORITHM:

1) ALL THE SELLERS SET A PRICE OF ZERO
2) DERIVE THE PREFERRED GRAPH AND CHECK WHETHER THERE IS A PERFECT MATCHING
3) IF YES, DONE
4) OTHERWISE, INCREASE BY ONE THE PRICE OF EACH SELLER WITH MORE THAN ONE PREFERENCE
5) REDUCE ALL THE PRICES BY THE SAME VALUE TO HAVE THE SMALLEST PRICE EQUAL TO ZERO
6) GO TO 2)

PROPERTY: MARKET-CLEARING PRICES LEAD TO MAXIMIZATION OF THE SOCIAL WELFARE (INCLUDING SELLERS AND BUYERS)

EXAMPLE FROM REAL-WORLD APPLICATIONS

NETFLIX IS AN EXAMPLE OF MATCHING MARKET



EACH CUSTOMER HAS PREFERENCES OVER THE DVDs AND, USUALLY, THE NUMBER OF AVAILABLE COPIES OF A DVD IS NOT SUFFICIENT

$\Downarrow$

BEST MATCH CONSIDERS ALSO THE SECOND, THIRD, ... PREFERENCE