A LITTLE RECAP ⊛

QUASI-LINEAR ENVIRONMENT: a quasi linear environment is characterized by:

- outcome space $X = \{(k, p_1, \dots, p_n) : k \in K,\ p_i \in \mathbb{R}\}$
- utility function $U_i(x, \theta_i) = U_i((k, p_1, \dots, p_n), \theta_i) = v_i(k, \theta_i) - p_i$, $v_i : K \times \Theta_i \to \mathbb{R}$

K is the set of allocations. $v_i(k, \theta_i)$ represent the evaluation of allocation k for i when her type is θ_i. p_i is the monetary payment of i to the mechanism, i.e., the auctioneer.

⊛ MECHANISM DESIGN: also known as "reverse game theory": the goal is to design a mechanism, a game, to satisfy a given function.
informally: we have to create a game, its rules, s.t. when others will play, we can drive them towards the goal. Here our perspective it's not of one of the players, we are the designer, e.g., the auctioneer in an auction.

ALLOCATION EFFICIENCY: a social choice function $f(\theta) = (k(\theta), p_1(\theta), \dots, p_n(\theta))$, where $k(\theta)$ is the allocation function, is ALLOCATIVELY EFFICIENT if $k(\theta)$ is defined as

$$k(\theta) \in \arg\max_{k \in K} \sum_{i \in N} v_i(k, \theta_i)$$

informally: we are maximizing the summation of the values of the players.

WEAK BUDGET BALANCE: a social function $f(\theta) = (k(\theta), p_1(\theta), \dots, p_n(\theta))$ is weakly budget balanced if:

$$\sum_{i \in N} p_i(\theta) \geq 0, \quad \forall \theta \in \Theta$$

Informally: if a social choice function is WBB, then the auctioneer will always have a non-negative revenue. Remember that payments are from the players to the mechanism!

GROVES MECHANISM: a direct revelation economic mechanism $(\Theta_1, \dots, \Theta_n, X, f)$ (in which $f(\theta) = (k(\theta), p_1(\theta), \dots, p_n(\theta))$ is a Groves mechanism if:

$$k \in \arg\max_{k' \in K} \sum_{i \in N} v_i(k', \theta_i)$$

$$p_i(\theta) = h_i(\theta_{-i}) - \sum_{j \in N \setminus \{i\}} v_j(k(\theta), \theta_j), \quad \forall i \in N,$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function on θ_{-i}

Given a social choice function f, i.e., $f : \Theta_1 \times \dots \times \Theta_n \to X$ assigns an outcome x to each possible profile of players' type, a DIRECT REVELATION ECONOMIC MECHANISM is a mechanism $(\Theta_1, \dots, \Theta_n, X, f)$
Remember that an economic mechanism is a tuple $(A_1, \dots, A_n, X, g)$ where A_i are actions of i, X is the set of outcomes and $g : A_1 \times \dots \times A_n \to X$ is the outcome function.

We can extend such mechanism as

VCG MECHANISM: a direct revelation economic mechanism $(\Theta_1, \dots, \Theta_n, X, f)$ in which $f(\theta) = (k(\theta), p_1(\theta), \dots, p_n(\theta))$ is a Clarke mechanism if:

$$k(\theta) \in \arg\max_{k' \in K} \sum_{i \in N} v_i(k', \theta_i)$$

$$p_i(\theta) = \max_{k' \in K_{-i}} \sum_{j \in N \setminus \{i\}} v_j(k', \theta_j) - \sum_{j \in N \setminus \{i\}} v_j(k(\theta), \theta_j), \quad \forall i \in N$$

where K_{-i} is the set of allocations when i is not present.

NO-SINGLE-AGENT EFFECT: a mechanism has no-single-agent effect if for every player i and for every profile of types θ there is a $k' \in K_{-i}$ s.t.

$$\sum_{j \in N \setminus \{i\}} v_j(k', \theta_j) \geq \sum_{j \in N \setminus \{i\}} v_j(k(\theta), \theta_j), \quad k(\theta) = \text{optimal allocation}$$

- If no-single-agent effect holds, then VCG is WBB.

VCG EXAMPLES

- Single-item auction (single type)

	k_1	k_2	k_3	k_4
1	θ_1			
2		θ_2		
3			θ_3	
4				θ_4

$\theta_1 = 0.8$

$\theta_2 = 0.6$

$\theta_3 = 0.2$

$\theta_4 = 0.5$

$p_1 = SW_{-1}(k^*_{-1}) - SW_{-1}(k^*) = 0.6 - 0$

$p_2 = SW_{-2}(k^*_{-2}) - SW_{-2}(k^*) = 0.8 - 0.8$

$p_3 = p_4 = p_2$

- Double-item auction (multiple type)

$\theta_1 = (\theta_{1.1}, \theta_{1.2}, \theta_{1.3}), \quad \theta_2 = (\theta_{2.1}, \theta_{2.2}, \theta_{2.3})$

	k_1 (1,1)	k_2 (1,2)	k_3 (2,1)	k_4 (2,2)
1	$\theta_{1.1}$	$\theta_{1.2}$	$\theta_{1.3}$	0
2	0	$\theta_{2.3}$	$\theta_{2.2}$	$\theta_{2.1}$

$\theta_1 = (0.6, 0.3, 0.2)$

$\theta_2 = (0.5, 0.3, 0.4)$

$k^* = k_2, \ SW(k_2) = 0.7$

	k_1	k_2	k_3	k_4
1	0.6	0.3	0.2	0
2	0	0.4	0.3	0.5

$p_1 = SW_{-1}(k^*_{-1}) - SW_{-1}(k^*) = 0.5 - (0.7 - 0.3) = 0.5 - 0.4 = 0.1$

$p_2 = SW_{-2}(k^*_{-2}) - SW_{-2}(k^*) = 0.6 - (0.7 - 0.4) = 0.6 - 0.3 = 0.3$

WEIGHTED VCG

If the evaluations of the players can be of any kind, this is the most general mecani[sm] that can be adopted.

$v_i(k, \theta_i) = \theta_{i,k}$, i.e., we can have one parameter for every allocation

We define the best allocation and payments as follows.

$w_i \in \mathbb{R}^+$: weight, defined for every player
$c_k \in \mathbb{R}$: cost, defined for every allocation
] application dependent

$$k(\theta) = \arg\max_{k' \in K} \left\{ \sum_{i \in N} w_i v_i(k', \theta) + c_{k'} \right\}$$

$$p_i(\theta) = \frac{h_i}{w_i}(\theta_{-i}) - \sum_{j \in N \setminus \{i\}} \frac{w_j}{w_i} v_j(k(\theta), \theta_j) - \frac{c_{k(\theta)}}{w_i},$$

where $h_i(\theta_{-i}) = \max_{k' \in K_{-i}} \sum_{j \in N \setminus \{i\}} w_j v_j(k', \theta_j)$

It's a VCG with weights for players. Such weights can be ≥ 1: they are normalized by the formula

Single-item auction

	k_1	k_2	k_3	k_4
1	θ_1			
2		θ_2		
3			θ_3	
4				θ_4

w_i
3
7
12
5

c_k
2
1
5
3

$\theta_1 = 0.8$; $\theta_2 = 0.6$; $\theta_3 = 0.2$; $\theta_4 = 0.5$

$SW(k_1) = 0.8 \cdot 3 + 2 = 4.4$

$SW(k_2) = 0.6 \cdot 7 + 1 = 5.2$: $k(\theta) = k_3$

$SW(k_3) = 0.2 \cdot 12 + 5 = 7.4$

$SW(k_4) = 0.5 \cdot 5 + 3 = 5.5$

$p_3 = \frac{5.5}{12} - 0 - \frac{5}{12} = \frac{0.5}{12} = \frac{1}{24} = 0.041\overline{6}$

$p_1 = p_2 = p_4 = 0$

Double-item auction

	k_1	k_2	k_3	k_4
1	$\theta_{1,1}$	$\theta_{1,2}$	$\theta_{1,3}$	0
2	0	$\theta_{2,3}$	$\theta_{2,2}$	$\theta_{2,1}$

w_i
5
10

c_k
2
1
5
2.5

$\theta_1 = (0.6, 0.3, 0.2)$

$\theta_2 = (0.5, 0.3, 0.4)$

$SW(k_1) = 0.6 \cdot 5 + 2 = 5$

$SW(k_2) = 0.3 \cdot 5 + 0.4 \cdot 10 + 1 = 6.5$: $k(\theta) = k_3$

$SW(k_3) = 0.2 \cdot 5 + 0.3 \cdot 10 + 5 = 9$

$SW(k_4) = 0.5 \cdot 10 + 2.5 = 7.5$

$p_1 = \frac{8}{5} - \frac{10}{5} \cdot \overset{0,3}{8} - \frac{5}{5} = -\frac{77}{5}$

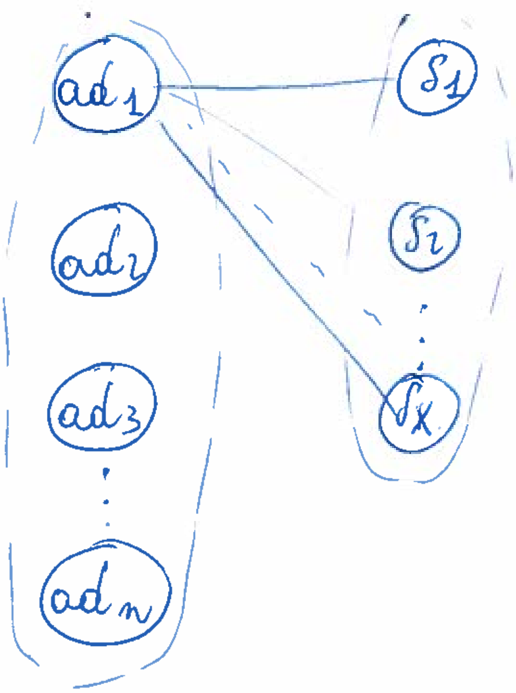
$p_2 = \frac{6}{10} - \frac{5}{10} \cdot \overset{0,2}{6} - \frac{5}{5} = -\frac{29}{5}$

: costs for allocations are too high!

VCG APPLICATIONS

SPONSORED SEARCH AUCTION (POSITION DEPENDENT)

There are n ads and x slots: k ads must be selected and ordered to be put in such slots.



There is a bipartite graph.
Each ad can be potentially chosen for each slot.

Each edge has a weight, i.e., the value obtained associating the ad to the slot.

$v_{i,\ell}$ = value to put ad_i in slot ℓ

$v_{i,\ell} = \Lambda_\ell q_i \theta_i$, Λ_ℓ = probability that the ad will be seen by the user

q_i = click-probability once the ad has been seen

θ_i = value given to the advertiser

Λ_ℓ and q_i are empirically estimated by the mechanism.
θ_i is the private information of the advertiser (its type)

The model of the user is Markovian: once an ad has been observed, there is a certain probability that it will look at the next one or go away.

VCG auction: we want to find the allocation $k(\theta) = \arg\max_{k' \in K} SW(k')$

payments: $p_i = \underbrace{SW_{-i}(k(\theta))}_{\text{allocation without } i} - \underbrace{SW_{-i}(k(\theta))}_{\text{remove bid of } i \text{ from optimal allocation}}$

In our case:

$$SW(K) = \sum_{J \in Ad}^{t} \Lambda_{Slot(J,K)} \, q_J \theta_J$$

$$SW_{-i}(K) = \sum_{J \in Ad \setminus \{i\}}^{t} \Lambda_{Slot(J,K)} \, q_J \theta_J$$

$\Lambda_i \quad q_i\theta_i$
$\vdots \quad\quad \vdots$
$\Lambda_J. \quad q_J\theta_J$

suppose $q_i\theta_i < q_J\theta_J$ and consider the swapped allocation. then the difference is:

$$(\Lambda_i q_i\theta_i + \Lambda_J q_J\theta_J) - (\Lambda_i q_J\theta_J + \Lambda_J q_i\theta_i) =$$
$$= \underbrace{(\Lambda_i - \Lambda_J)}_{>0}\underbrace{(q_i\theta_i - q_J\theta_J)}_{<0} :$$ not greedy is worse!

$K(\theta)$ orders in descendent way by q multiplied by θ. We take the first K and put in the slots.

Let's put some numbers (all values are in $[0,1]$ for simplicity)

	q_i	θ_i	$q_i\theta_i$
ad_1	$1/2$	1	$1/2$
ad_2	$1/3$	$1/2$	$1/6$
ad_3	$3/4$	$3/4$	$9/16$
ad_4	1	$1/4$	$1/4$

We have 4 ads but only 3 slots.

$\Lambda_1 = 1$
$\Lambda_2 = 0{,}9$
$\Lambda_3 = 0{,}7$

Optimal allocation: $K(\theta) = \langle ad_3, ad_1, ad_4, ad_2 \rangle$

$$SW(x^*) = \frac{9}{16} \cdot 1 + \frac{1}{2} \cdot 0{,}9 + \frac{1}{4} \cdot 0{,}7 + \frac{1}{6} \cdot 0 = \frac{19}{16} \qquad x^* = K(\theta)$$

$$SW_{-1}(x^*_{-1}) = SW_{-1}(\langle ad_3, ad_4, ad_2, ad_1 \rangle) = \frac{9}{16} \cdot 1 + \frac{1}{4} \cdot 0{,}9 + \frac{1}{6} \cdot 0{,}7 + \frac{1}{2} \cdot 0 = \frac{217}{240}$$

$$SW_{-2}(x^*_{-2}) = SW_{-2}(\langle ad_3, ad_1, ad_4, ad_2 \rangle) = \frac{9}{16} \cdot 1 + \frac{1}{2} \cdot 0{,}9 + \frac{1}{4} \cdot 0{,}7 + \frac{1}{6} \cdot 0 = \frac{19}{16}$$

$$SW_{-3}(x^*_{-3}) = SW_{-3}(\langle ad_1, ad_4, ad_2, ad_3 \rangle) = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 0{,}9 + \frac{1}{6} \cdot 0{,}7 + \frac{9}{16} \cdot 0 = \frac{101}{120}$$

$$SW_{-4}(x^*_{-4}) = SW_{-4}(\langle ad_3, ad_1, ad_2, ad_4 \rangle) = \frac{9}{16} \cdot 1 + \frac{1}{2} \cdot 0{,}9 + \frac{1}{6} \cdot 0{,}7 + \frac{1}{4} \cdot 0 = \frac{271}{240}$$

$SW_{-1}(x^*) = \frac{59}{80}$; $SW_{-2}(x^*) = \frac{19}{16}$; $SW_{-3}(x^*) = \frac{5}{8}$; $SW_{-4}(x^*) = \frac{81}{80}$

$$P_1 = SW_{-1}(x^*_{-1}) - SW_{-1}(x^*) = \frac{217}{240} - \frac{59}{80} = \frac{1}{6}$$
$$P_2 = SW_{-2}(x^*_{-2}) - SW_{-2}(x^*) = \frac{19}{16} - \frac{19}{16} = 0$$
$$P_3 = SW_{-3}(x^*_{-3}) - SW_{-3}(x^*) = \frac{101}{120} - \frac{5}{8} = \frac{13}{60}$$
$$P_4 = SW_{-4}(x^*_{-4}) - SW_{-4}(x^*) = \frac{271}{240} - \frac{81}{80} = \frac{7}{60}$$

$$\rightarrow \frac{1}{6} + 0 + \frac{13}{60} + \frac{7}{60} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Revenue for the auctioneer

Payments in SSA are made when an ad is clicked: $\mathbb{E}[p_i^c] = p_i$
Thus, real payments are:

$$p_i^c = \begin{cases} \frac{p_i}{q_i \Lambda_{slot(i,k)}} & , \text{ if ad}_i \text{ is clicked} \\ 0 & , \text{ otherwise} \end{cases}$$

In our example:

$$p_1^c = \frac{1}{6}\left(\frac{1}{\frac{1}{2}\cdot\frac{9}{10}}\right) = \frac{20}{6\cdot 9} = \frac{10}{27} \quad ; p_2^c = 0$$

$$p_3^c = \frac{13}{60}\cdot\left(\frac{1}{\frac{3}{4}\cdot 1}\right) = \frac{13}{60}\cdot\frac{4}{3} = \frac{13}{45} \quad ; p_4^c = \frac{7}{60}\cdot\left(\frac{1}{1\cdot\frac{7}{10}}\right) = \frac{7}{60}\cdot\frac{10}{7} = \frac{1}{6}$$

APX-SSA

Let us consider the following problem.
In general, Λ_j are known by the search engine since the mechanism has been used for a lot of time. But what about q_i? What happens when the mechanism is started? And what happens if a new advertiser enters the market?
q_i are not known, they are estimated as $\tilde{q}_i$, which can also evolve in time.
What happens if a mechanism cannot use q_i but just the estimated $\tilde{q}_i$?

$k(\theta) = \arg\max_{k' \in K} \sum'_{i \in N} q_i \theta_i \Lambda_{slot(i,k')}$: this is what we would like to have

Actually, we have only $\tilde{q}_i$: $\tilde{k}(\theta) = \arg\max_{k' \in K} \sum'_{i \in N} \tilde{q}_i \theta_i \Lambda_{slot(i,k')}$

$$\tilde{p}_i(\theta) = \max_{k' \in K} \sum'_{j \in N\setminus\{i\}} \tilde{q}_j \theta_j \Lambda_{slot(j,k')} - \sum'_{j \in N\setminus\{i\}} \tilde{q}_j \theta_j \Lambda_{slot(j,\tilde{k}(\theta))}$$

$$\tilde{p}_i^c = \begin{cases} \frac{\tilde{p}_i(\theta)}{\tilde{q}_i \Lambda_{slot(i,\tilde{k}(\theta))}} & , \text{ if ad}_i \text{ is clicked} \\ 0 & , \text{ otherwise} \end{cases}$$

What can we say about $\tilde{k}, \tilde{p}_i$? Is such mechanism incentive compatible?
In fact, we adopted the VCG mechanism for SSA because we had real q_i, now we have just approximations.
What about a WVCG? If it is, then it will be incentive compatible.

To prove this, we have to show that allocation and payments computed by $\tilde{k}(\theta)$, $\tilde{p}_i^c(\theta)$ are the same that would be computed by WVCG.

$$k(\theta) = \arg\max_{k' \in K} \left\{ \sum_{i \in N}{}' w_i q_i \theta_i \Lambda_{slot(i,k')} + c_{k'} \right\} = \arg\max_{k' \in K} \left\{ \sum_{i \in N} \tilde{q}_i \theta_i \Lambda_{slot(i,k')} \right\} = \tilde{k}(\theta)$$

$$w_i = \frac{\tilde{q}_i}{q_i}$$

$$c_{k'} = 0$$

$$p_i(\theta) = \frac{1}{w_i} h_i(\theta_{-i}) - \frac{1}{w_i} \sum_{j \in N \setminus \{i\}} w_j q_j \theta_j \Lambda_{slot(j,k(\theta))} = \left[w_i = \frac{\tilde{q}_i}{q_i} \right]$$

$$= \frac{q_i}{\tilde{q}_i} \left(\sum_{j \in N \setminus \{i\}}{}' \tilde{q}_j \theta_j \Lambda_{slot(j,k')} - \sum_{j \in N \setminus \{i\}} \tilde{q}_j \theta_j \Lambda_{slot(j,k(\theta))} \right)$$

There is q_i but real payments are in the clicks!

We compute the expected value on $\tilde{p}_i^c$:

$\mathbb{E}[\tilde{p}_i^c(\theta)] = p_i(\theta)$? This is what we want: if this is true, our payments are the same computed by the WVCG.

$$\mathbb{E}[\tilde{p}_i^c(\theta)] = \frac{\tilde{p}_i(\theta)}{\tilde{q}_i \Lambda_{slot(i,\tilde{k}(\theta))}} \cdot P[\text{click}] = \frac{\tilde{p}_i(\theta)}{\tilde{q}_i \Lambda_{slot(i,\tilde{k}(\theta))}} \cdot \Lambda_{slot(i,\tilde{k}(\theta))} \cdot q_i =$$

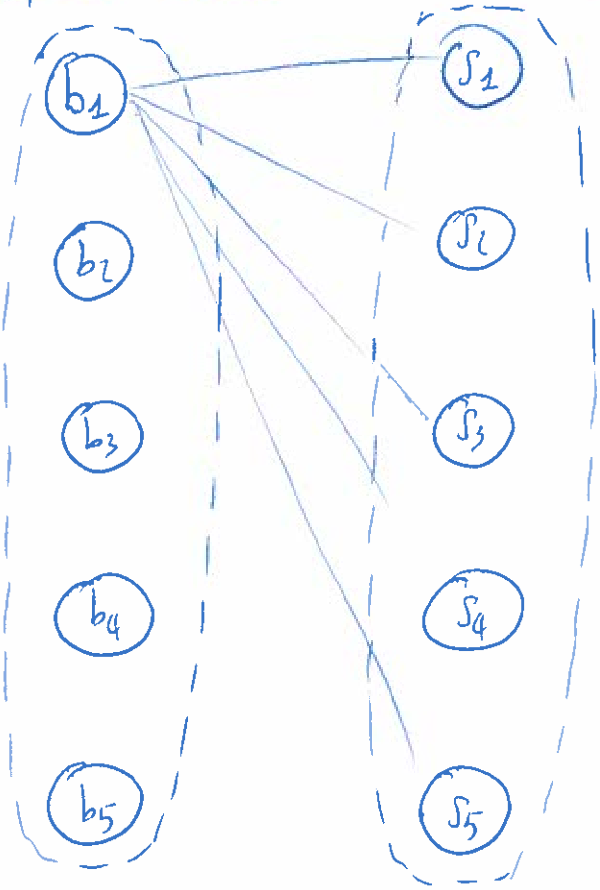
if the ad is seen, the click is real

$$= \frac{\tilde{p}_i(\theta)}{\tilde{q}_i} \cdot q_i = \left[\tilde{p}_i(\theta) = w_i \, p_i(\theta) \right] = \frac{q_i}{\tilde{q}_i} \cdot \frac{\tilde{q}_i}{q_i} \cdot p_i(\theta) = p_i(\theta)$$

<u>N.B.</u>: for <u>every</u> $\tilde{q}_i$, the mechanism is incentive compatible.
It doesn't matter how 'wrong' is the starting estimate, the mechanism will be truthful anyway!

DOUBLE AUCTION

There are two groups: sellers s_i and buyers b_j (for simplicity, the number of buyers is equal to the number of sellers).
We have to match buyers and sellers.
The main difference w.r.t. SSA is that on both sides there is some private information, i.e., types of buyers and sellers while before we had just the type of the ads.
Thus, the value is associated with private info on both sides.



Also here we have a bipartite graph.

$v_{ij} = \theta_{b_i} - \theta_{s_j}$: θ_{b_i} = budget available to b_i

θ_{s_j} = cost of the item sold by s_j

We want to apply the VCG:

allocation $X^* = \{(b_i, s_j), (b_k, s_k), \dots\}$

$$X^* \in \underset{X}{\text{ARG MAX}}\ SW(X) = \sum_i (\theta_{b_i} - \theta_{s_i})$$

$$p_i = SW_{-i}(X^*_{-i}) - SW_{-i}(X^*)$$

How can we compute X^*?
We order buyers in descending order and sellers in ascending order.
Then we match b_i and s_i if and only if $b_i \geq s_i$

<u>N.B.</u>: the goal of the auctioneer is to maximize its revenue, not to match all buyers and sellers!

Time required to compute X^*: $2n \log n$ to order b_i and s_i
n comparisons

$$O(2n \log n + n)$$

Adopting the VCG, the revenue of the auctioneer is always not positive. (1)
Since the 'no single agent effect' does not hold, the weak budget balance property does not hold anymore.

No single agent effect: a mechanism is no-single-agent effect if for every player i and for every profile of types θ there is a $k' \in K_{-i}$ s.t.: $\sum'_{j \in N\setminus\{i\}} v_j(k', \theta_j) \geq \sum'_{j \in N\setminus\{i\}} v_j(k(\theta), \theta_j)$

WBB: $\sum'_{i \in N} p_i(\theta) \geq 0, \forall \theta \in$ (1)

Let's put some numbers.

$\theta_{b_1} = 10$; $\theta_{b_2} = 9$; $\theta_{b_3} = 7$; $\theta_{b_4} = 5$; $\theta_{b_5} = 3$

$\theta_{s_1} = 0$; $\theta_{s_2} = 3$; $\theta_{s_3} = 5$; $\theta_{s_4} = 6$; $\theta_{s_5} = 7$

$x^* = \{(b_1, s_1), (b_2, s_2), (b_3, s_3)\}$: $SW(x^*) = v_{11} + v_{22} + v_{33} = 10 + 6 + 2 = 18$

Now we compute payments both for buyers and for seller.

$$p_{b_1} = ((\theta_{b_2} - \theta_{s_1}) + (\theta_{b_3} - \theta_{s_2})) - (SW(x^*) - \theta_{b_1}) = (9+4) - (18-10) = 5$$
$$p_{b_2} = ((\theta_{b_1} - \theta_{s_1}) + (\theta_{b_3} - \theta_{s_2})) - (SW(x^*) - \theta_{b_2}) = (10+4) - (18-9) = 5$$
$$p_{b_3} = ((\theta_{b_1} - \theta_{s_1}) + (\theta_{b_2} - \theta_{s_2})) - (SW(x^*) - \theta_{b_3}) = (10+6) - (18-7) = 5$$
$$p_{b_4} = p_{b_5} = 0$$

+ 15 for the auctioneer

$$p_{s_1} = ((\theta_{b_1} - \theta_{s_2}) + (\theta_{b_2} - \theta_{s_3}) + (\theta_{b_3} - \theta_{s_4})) - (SW(x^*) - \theta_{s_1}) = (7+4+1) - 18 = -6$$
$$p_{s_2} = ((\theta_{b_1} - \theta_{s_1}) + (\theta_{b_2} - \theta_{s_3}) + (\theta_{b_3} - \theta_{s_4})) - (SW(x^*) - \theta_{s_2}) = (10+4+1) - (18+3) = -6$$
$$p_{s_3} = ((\theta_{b_1} - \theta_{s_1}) + (\theta_{b_2} - \theta_{s_2}) + (\theta_{b_3} - \theta_{s_4})) - (SW(x^*) - \theta_{s_3}) = (10+6+1) - (18+5) = -6$$
$$p_{s_4} = p_{s_5} = 0$$

- 18 for the auctioneer

revenue for auctioneer: $15 - 18 = -3$

So, what does this mean? VCG is not the right way to deal with a double auction.
We'll see one of the next times the best approach.