Equilibria and Stability in A Model of Political and Social Conflict.

The Symmetric Case

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VERY PRELIMINARY AND INCOMPLETE

The structure of the presentation

1. Introduction:
    1. stylized facts
    2. the general research program
    3. specific questions addressed in this paper
    4. reference literature
    5. the contribution of the paper
2. The model
3. The cases
4. Interpretation of the results
5. Conclusion

Introduction - 1
Stylized facts

- Real conflicts (wars, political, social, interpersonal) are characterized by
  1. sudden surge
  2. huge diversity in evolution through time and in outcomes.
- Why?
- Two possible explanations:
  1. structural change in the environment $\Leftrightarrow$ comparative statics, which "*requires*" uniqueness of equilibria
  2. dynamics out of equilibrium $\Leftrightarrow$ stability, which "*requires*" multiplicity of equilibria to be interesting

Introduction - 2
Research Program - 1

GENERAL PROBLEM: Why actual conflicts present so many different situations and evolution through time?
Can we provide an explanation of the two common characteristics of actual conflicts, i.e. that

1. very often they came as a surprise
2. their evolution through time is very heterogenous, sometimes they quickly disappear sometimes they degenerate in catastrophe

Introduction - 3
Research Program - 2

SPECIFIC PROBLEM:

What are the effects of different

1. conflict technologies
2. costs functions
3. agents goals on
    1. Set of equilibria (unique or multiple)
    2. comparative statics
    3. dynamics out of equilibria (stability and cycles)

In a perfectly symmetric game with two players.

Introduction - 4
Specific questions addressed in this paper

- The main focus of this work is to explore the joint work of
  1. conflict technologies
  2. costs
  3. polarization

on the

1. relation between this model and standard conflict models
2. existence of asymmetric and multiple equilibria
3. evolution of conflicts out of equilibrium according to the best reply dynamics
4. characterization of basins of attraction.

Introduction - 5
Reference literature

 Corchon Luis C. 2007. The theory of contests: a survey. Review of Economic Design,

 Garfinkel, Michelle R. and Stergios Skaperdas 2007. Economics of Conflict: An Overview.

 Konrad Kai A. 2009. Strategy and Dynamics in Contests.

 Chowdhury Subhasish M. and Roman M. Sheremeta 2011a. A generalized Tullock contest. Public Choice, 147: 413-420.

 Chowdhury Subhasish M. and Roman M. Sheremeta 2011b. Multiple equilibria in Tullock contests. Economic Letters, 112: 216-219.

 Szidarovszky, F. and Koji O., 1997. On the existence and uniqueness of pure Nash equilibrium in rent-seeking. Games and Economic Behavior.

Introduction - 6
The contribution of the paper - 1

1. we qualify the characteristics of our model and its properties with respect to the literature on contests
2. For the case of simple conflict technology without externalities and
    1. linear/concave/convex cost function with or without fixed costs

we provide a characterization of

i. equilibrium
ii. comparative statics
iii. stability.

Introduction - 7
The contribution of the paper - 2

3. For the case of simple technology with externalities and
   1. linear cost function with or without fixed costs

we provide a qualitative characterization of

i. equilibrium
ii. comparative statics
iii. stability

4. For the case of simple conflict technology without externalities and linear cost function and
   1. polarizion with externalities

we provide a characterization of

i. equilibrium
ii. comparative statics
iii. stability.

The Model - 1

Definition

A Model of Political and Social Conflict is defined by the following element:

1. *two agents*, part 1 and 2, denoted by

$$i, j \in \{1, 2\};$$

2. the agents' *efforts*:

$$x_i \in X_i \subseteq \mathbb{R}^+, \ i \in \{1, 2\};$$

3. the agents' *effectiveness of efforts*:

$$S_i(x_i, x_j) : X_i \times X_j \to \mathbb{R}^+, \ i \in \{1, 2\};$$

The Model - 2

Definition

4. the agents' *goals*:

$$g_i : X_i \times X_j \to \left[\underline{g_i}, \overline{g_i}\right] \subseteq \mathbb{R}, \ i \in \{1, 2\};$$

$$\theta\left(x_i, x_j\right) := g_2\left(x_2, x_1\right) - g_1\left(x_1, x_2\right) > 0$$

is a measure of social and political *polarization;*

5. the *outcome function* describes the result of the conflict as a function of both agents' efforts' effectiveness

$$\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to \left[g_1, g_2\right];$$

as ζ is near g_2, we have that 2 is prevailing, while if ζ is near g_1 it states the superiority of 1;

The Model - 3

Definition

6. the *conflict technology* connecting agents' effectiveness of the efforts $S_i(x_i, x_j)$ to the probability of getting an outcome:
$$P : \mathbb{R}^+ \times \mathbb{R}^+ \to \Delta([g_1, g_2]),$$
7. the agents' *utility function* evaluating the (possibly random) outcomes:
$$U_i : [g_1, g_2] \to \mathbb{R}^+, \ i \in \{1, 2\};$$
8. the agents' *cost function*
$$C_i(x_i, x_j) : X_i \times X_j \to \mathbb{R}^+, i \in \{1, 2\}.$$

Observations on the Model

Remark

1. *If the set of possible player's effort X_i is bounded, then wlg we can normalize the intensity of the effort $x_i \in [0, 1]$.*
2. *If we assume that the agents' goals g_i do not depend on the players' efforts, then we put our model within the class of rent seeking models, while in the models of production and conflict the value of the goals is endogenously determined by the agents' choices*
3. *the separation between utility and cost functions is a common restriction in the literature of economics of information.*

The Model - 4

Consider the standard definition of Contest, e.g. Corchon 2007:

Definition

A Contest is defined by the following element:

1. A finite set of *agents*, the contenders:

$$N = \{1, ..., n\};$$

2. A set of possible *actions* taken by the agents before a prize is allocated:

$$a_i \in A_i \ \forall i \in N;$$

3. A *prize* that may depend on agents' actions:

$$V_i : A_1 \times ... \times A_n \to \mathbb{R} \ \forall i \in N;$$

let denote the set of i's possible prizes as Ω_i;

The Model - 5

Definition

4. A *contest success function* relating agents' actions to the probability of obtaining the prize:

$$p_i : A_1 \times ... \times A_n \rightarrow \Delta(\Omega_i) \ \ \forall i \in N;$$

5. A *utility function* evaluating each agent prize:

$$U_i : \Omega_i \rightarrow \mathbb{R} \ \forall i \in N;$$

6. A *cost function* relating agents' actions to the cost of the actions:

$$C_i : A_1 \times ... \times A_n \rightarrow \mathbb{R} \ \forall i \in N.$$

Remark

It is immediate to see that a Model of Political and Social Conflict is a particular specific Contest.

Remark

A peculiar difference between contests and MPSC is that in our model outcomes and conflict technology are objective, while subjectivity, i.e. agents' dependence, regard the effectiveness of effort and the preferences, a characteristic that we believe better match real conflicts, helping interpretation and understanding.

Agents' Utilities

Hypothesis

The agents' utility function is linearly decreasing in the distance d_i between its goal and the outcome:

$$d_1(x_1, x_2) = \zeta(S_i(x_i, x_j), S_j(x_j, x_i)) - g_1(x_1, x_2),$$

$$d_2(x_2, x_1) = g_2(x_2, x_1) - \zeta(S_i(x_i, x_j), S_j(x_j, x_i))$$

Then wlg we normalize the linear utility function, so that

$$U_i(x_i, x_j) = -d_i(x_i, x_j).$$

Remark

This functional form is useful to focus on the role of polarization.

Outcomes - 1

Hypothesis

the outcome of the conflict is either deterministic and equal to

$$\zeta\left(S_1, S_2\right) = \frac{S_1\left(x_1, x_2\right) g_1\left(x_1, x_2\right)}{S_1 + S_2} + \frac{S_2\left(x_2, x_1\right) g_1\left(x_1, x_2\right)}{S_1 + S_2},$$

with the following conflict technology

$$P\left(z \middle| S_1\left(x_1, x_2\right), S_2\left(x_2, x_1\right)\right) = \begin{cases} 1 & \text{if } z = \frac{S_1 g_1}{S_1+S_2} + \frac{S_2 g_2}{S_1+S_2} \\ 0 & \text{otherwise.} \end{cases}$$

Outcomes - 2

Hypothesis

or random

$$\zeta\left(S_1\left(x_1,x_2\right),S_2\left(x_2,x_1\right)\right)\in\left\{g_1\left(x_1,x_2\right),g_2\left(x_2,x_1\right)\right\}$$

with the following conflict technology

$$P\left(z|S_1,S_2\right)=\begin{cases}\frac{S_1(x_1,x_2)}{S_1(x_1,x_2)+S_2(x_2,x_1)} & \text{if } z=g_1\left(x_1,x_2\right)\\ \frac{S_2(x_2,x_1)}{S_1(x_1,x_2)+S_2(x_2,x_1)} & \text{if } z=g_2\left(x_2,x_1\right)\\ 0 & \text{otherwise.}\end{cases}$$

Outcomes - 3

Remark

1. *risk neutrality means that these two possibilities are strategically equivalent*
2. *the functional form of the weights or of the conflict technology when the outcome is random means that we are restricting ourself to the class of "Ratio Success Functions"*

Costs Functions

- In production theory cost functions are derived from technology and from a "reduced form" dynamics, i.e. the distinction between short and long run
- Fixed costs are a reduced form of the impact of past decisions on today decisions only if this past effects are redeemable
- Their application to conflict theory requires further considerations.

- In conflict's costs should be included not only the direct costs of exerting effort, including opportunity costs, but also this "reduced form" dynamics
- If we restrict ourself to direct conflict's costs, then all fixed costs are sunk, they can not be redeemed since usually there is no market for conflict's efforts
- Conflict's fixed costs might be used to model a "reduced form" of continuation payoff, which means that fixed costs might be negative when conflicting is seen as a way of establishing future positive reputation.

The reduced form of our MPSC

Proposition

Hypotheses 1 to 7 imply

$$E\left[\zeta\left(S_1\left(x_1,x_2\right),S_2\left(x_2,x_1\right)\right)\right]=$$

$$\begin{aligned}
&= \frac{S_2\left(x_2,x_1\right)}{S_1\left(x_1,x_2\right)+S_2\left(x_2,x_1\right)}\theta(x_1,x_2)+g_1(x_1,x_2)= \\
&= -\frac{S_1\left(x_1,x_2\right)}{S_1\left(x_1,x_2\right)+S_2\left(x_2,x_1\right)}\theta(x_1,x_2)+g_2(x_2,x_1),
\end{aligned}$$

hence

$$\pi_i(x_i,x_j)=-\frac{S_j\left(x_j,x_i\right)}{S_1\left(x_1,x_2\right)+S_2\left(x_2,x_2\right)}\theta\left(x_1,x_2\right)-C_i\left(x_i,x_j\right).$$

MPSC and contest

Remark

Hence, under these assumptions, a MPSC is equivalent to a contest where

1. *the* **contest success function** *is*

$$\frac{S_j\left(x_j, x_i\right)}{S_1\left(x_1, x_2\right) + S_2\left(x_2, x_1\right)}$$

2. *the* **prize** *is*

$$-\theta\left(x_1, x_2\right)$$

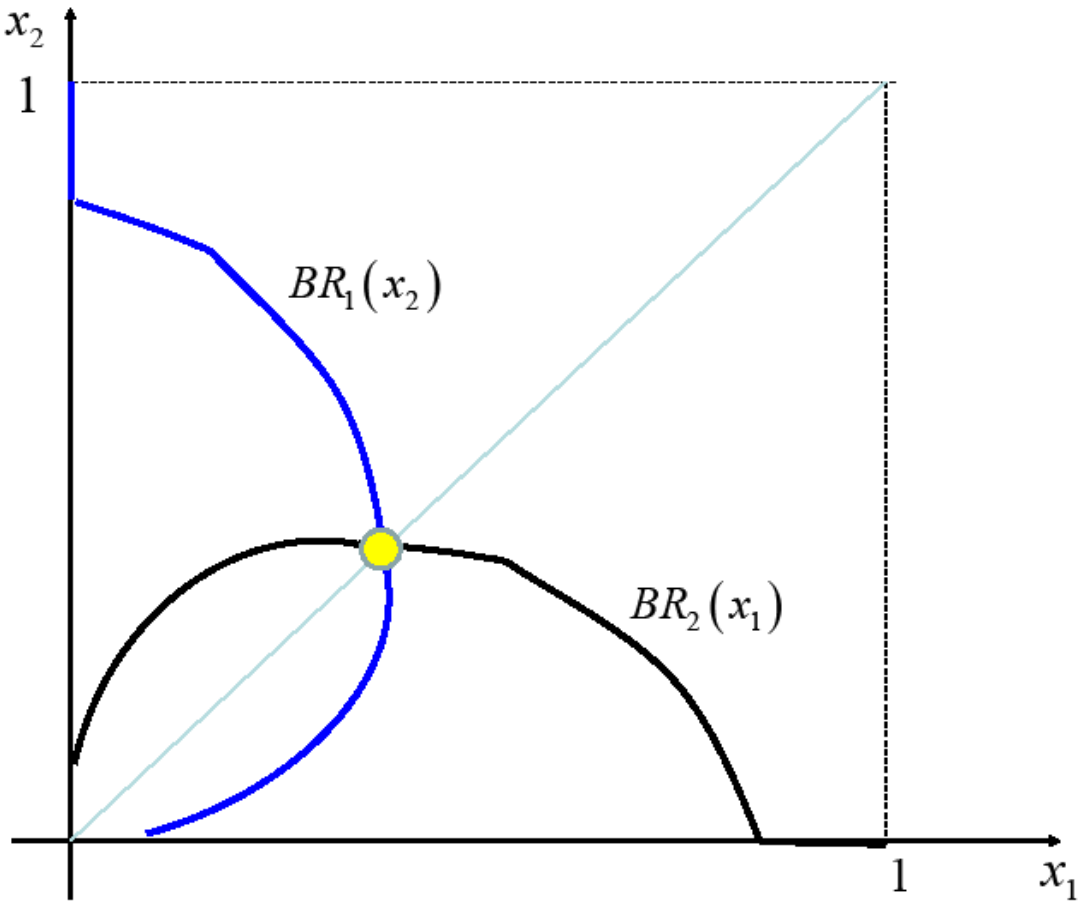
which means that from the point of view of the contest theory we are dealing with a very peculiar setting. In particular, note that our contest success function is different from the general technology used by Szidarovszky and Okuguchi 1997 which might look very similar.

Case 1: linear technology, linear costs and no externalities

Hypothesis

1. *Each agent's effectivity function is linear in the intensity of the effort:*
$$S(x_i, x_j) = \beta x_i + 1,$$
2. *Each agent's cost function is linear in the intensity of the effort :*
$$c_1 = 0, c_2 = c, c_3 \in \{0, K\}.$$
3. *Polarization is constant*
$$\theta(x_i, x_j) = \theta > 0.$$

The Best Replies in case 1 & zero fixed costs



The Equilibrium in case 1 & zero fixed costs

Proposition

EXISTENCE OF NASH EQUILIBRIA: *When conflict technologies and costs are linear without externalities and there are no fixed costs, then there exists a unique symmetric pure strategy Nash equilibrium such that*

$$\left(x_1^{NE}, x_2^{NE}\right) = \begin{cases} (0,0) & \text{if } \frac{\beta}{4c} \in \left[0, \frac{1}{\theta}\right] \\ \left(\frac{\theta}{4c} - \frac{1}{\beta}, \frac{\theta}{4c} - \frac{1}{\beta}\right) & \text{if } \frac{\beta}{4c} \in \left[\frac{1}{\theta}, \frac{1}{\theta - 4c}\right] \\ (1,1) & \text{if } \frac{\beta}{4c} \in \left[\frac{1}{\theta - 4c}, \infty\right) \end{cases}$$

Comparative statics in case 1 & zero fixed costs - 1

Proposition

COMPARATIVE STATIC: *When conflict technologies and costs are linear without externalities and there are no fixed costs, then*

1. *an increment in the marginal cost of conflicts c reduces the intensity of conflict in equilibrium,*
2. *an increase of the marginal productivity of the effort β on its effectivity increases the intensity of conflict in equilibrium*
3. *similarly, an increase in polarization θ increases the intensity of conflict in equilibrium.*

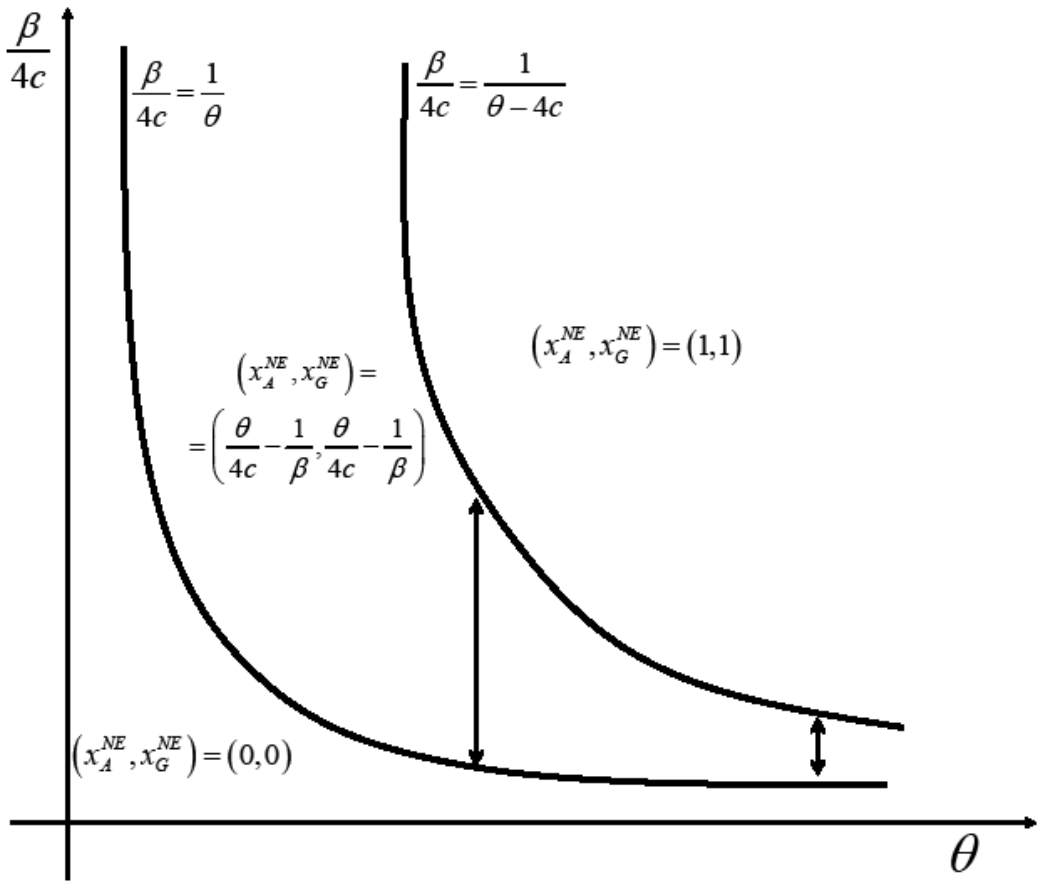
Comparative statics in case 1 & zero fixed costs - 2

Proposition

COMPARATIVE STATIC: *When conflict technologies and costs are linear without externalities and there are no fixed costs, then*

1. *an increase in the marginal cost of conflict c implies that the interval of β such that there is zero conflict is increasing, however has an ambiguous effect on the other intervals, since $\left[\frac{4c}{\theta}, \frac{4c}{\theta-4c}\right]$ is increasing but the condition $\theta \geq 4c$ is stronger while $\left[\frac{4c}{\theta}, +\infty\right)$ and $\left[\frac{4c}{\theta-4c}, +\infty\right)$ are both shrinking;*
2. *on the other hand an increase in polarization θ has the certain effect of shrinking the interval of β such that there is zero conflict and increasing all the other intervals.*

A Global View of the Equilibria in case 1 & zero fixed costs



Evolution of conflict in case 1 & zero fixed costs

Proposition

STABILITY: *When conflict technologies and costs are linear without externalities and there are no fixed costs, then all the three possible equilibria are fully stable wrt to the best reply dynamics, independently of* β, c, θ.

Equilibrium & Evolution with convex or concave costs & zero fixed costs

Proposition

EXISTENCE OF NASH EQUILIBRIA: *When conflict technologies are linear without externalities, costs are convex or concave and there are no fixed costs, then there exists a unique symmetric pure strategy Nash equilibrium such that either*

$$\left(x_1^{NE}, x_2^{NE}\right) = (0,0)$$

or

$$\left(x_1^{NE}, x_2^{NE}\right) \in (0,1) \times (0,1)$$

or

$$\left(x_1^{NE}, x_2^{NE}\right) = (1,1)\,.$$

Proposition

STABILITY: *When conflict technologies are linear without externalities, costs are convex or concave and there are no fixed costs, the possible equilibria are fully stable with respect to the best reply dynamics, independently of* β, c, θ.

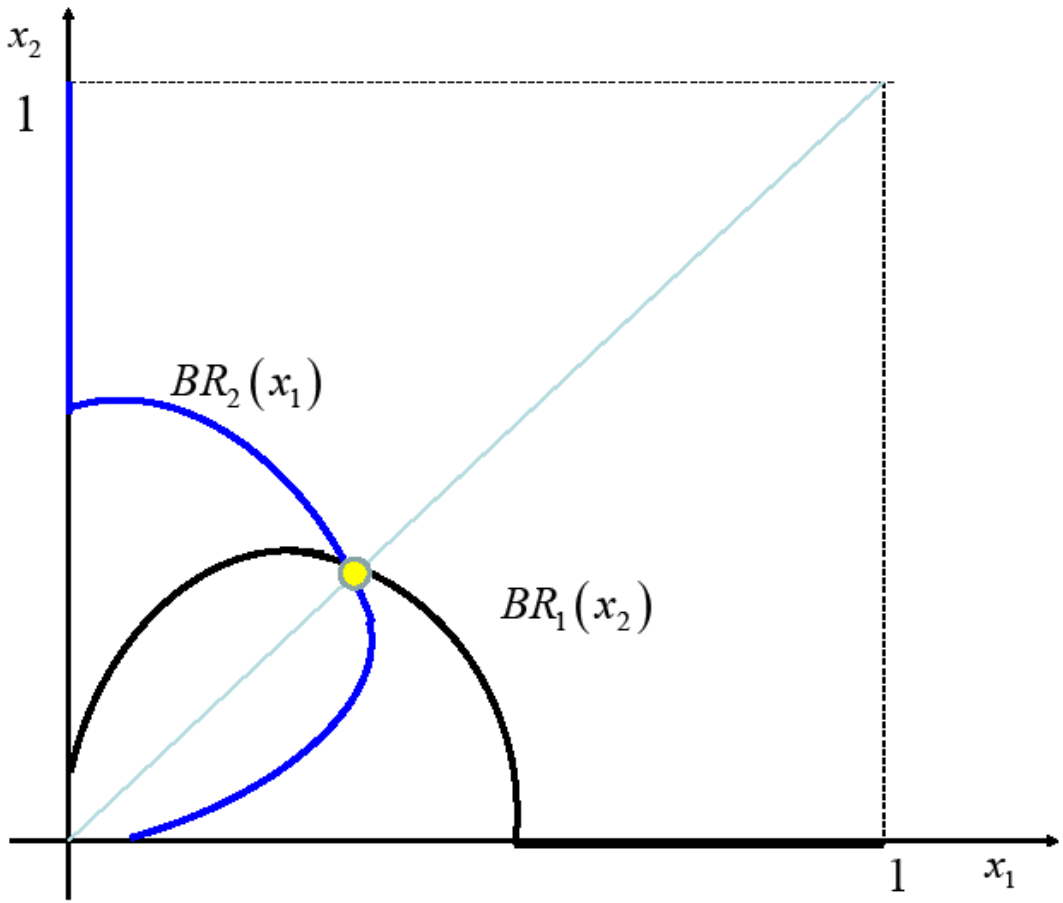
The Equilibrium with fixed costs

Proposition

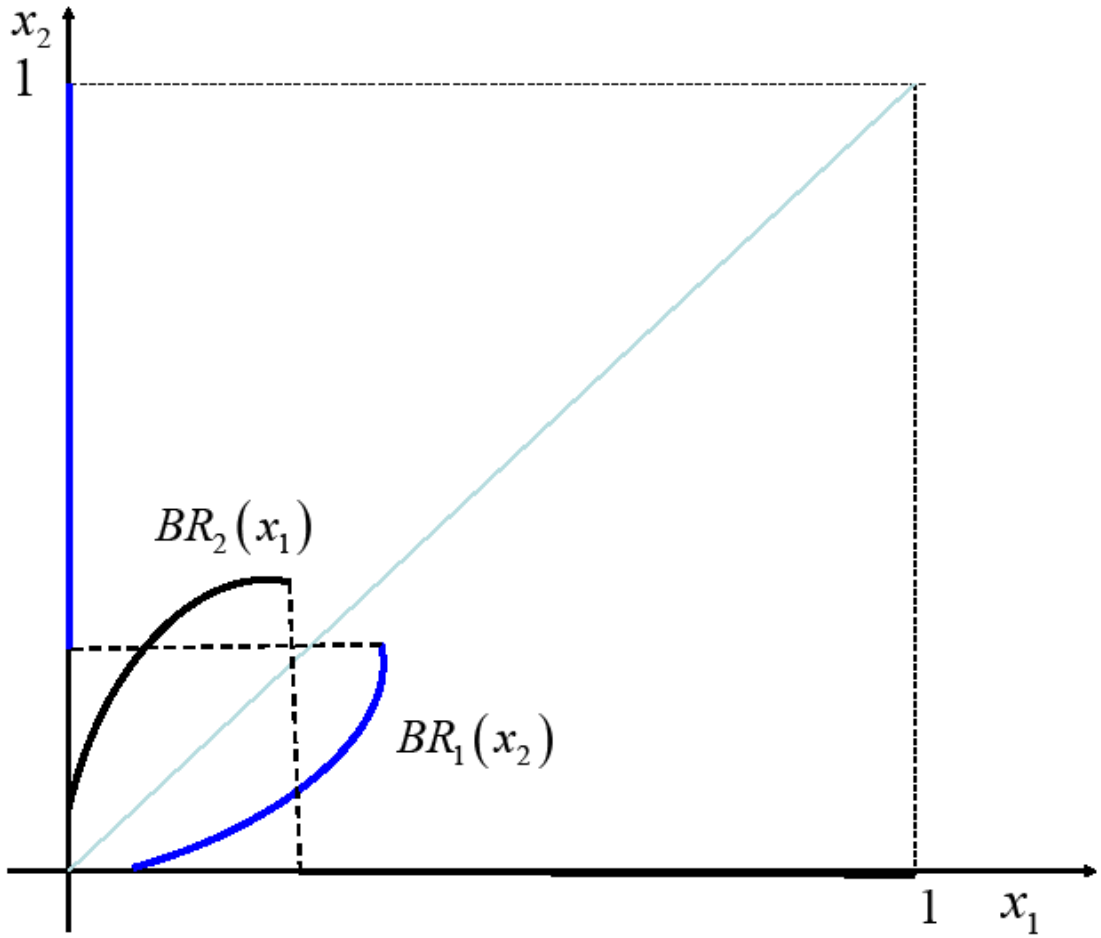
When conflict technologies and costs functions are linear without externalities and there are fixed costs, then

1. *when the fixed costs are small, i.e.* $K \in \left[0, \frac{\beta\theta^2}{\beta\theta+4c} + \frac{c}{\beta} - \frac{3}{4}\theta\right]$, *then they are negligible and we get the equilibria of case 1-A;*
2. *when the fixed costs are intermediate, i.e.* $K \in \left[\frac{\beta\theta^2}{\beta\theta+4c} + \frac{c}{\beta} - \frac{3}{4}\theta, Z(\beta,\theta,c)\right]$, *then there is no pure strategy equilibrium, hence the outcomes are random;*
3. *when the fixed costs are huge, i.e.* $K \in [Z(\beta,\theta,c), \infty]$, *then in equilibrium there is no conflict;*
4. *when the fixed costs are negative, then in equilibrium either they are irrelevant or they induce maximum conflict*

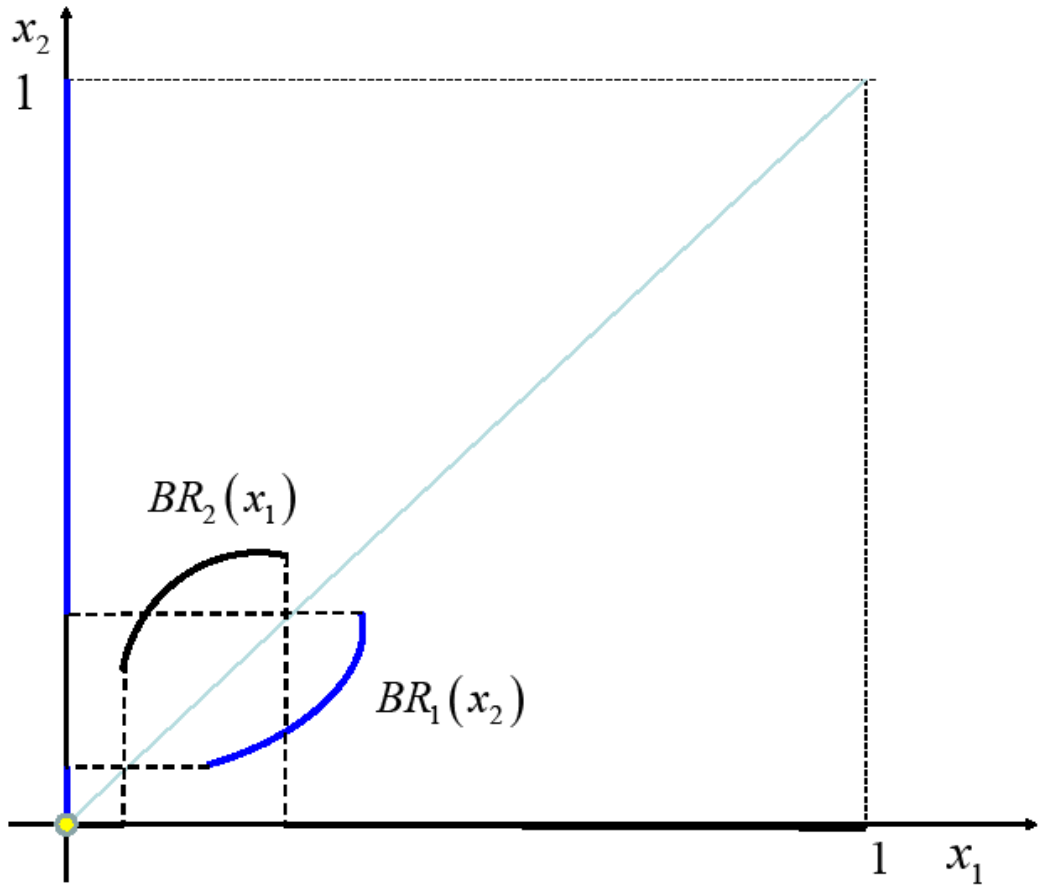
The Best Replies with linear costs & negligible fixed costs



The Best Replies with linear costs & intermediate fixed costs



The Best Replies with linear costs & huge fixed costs



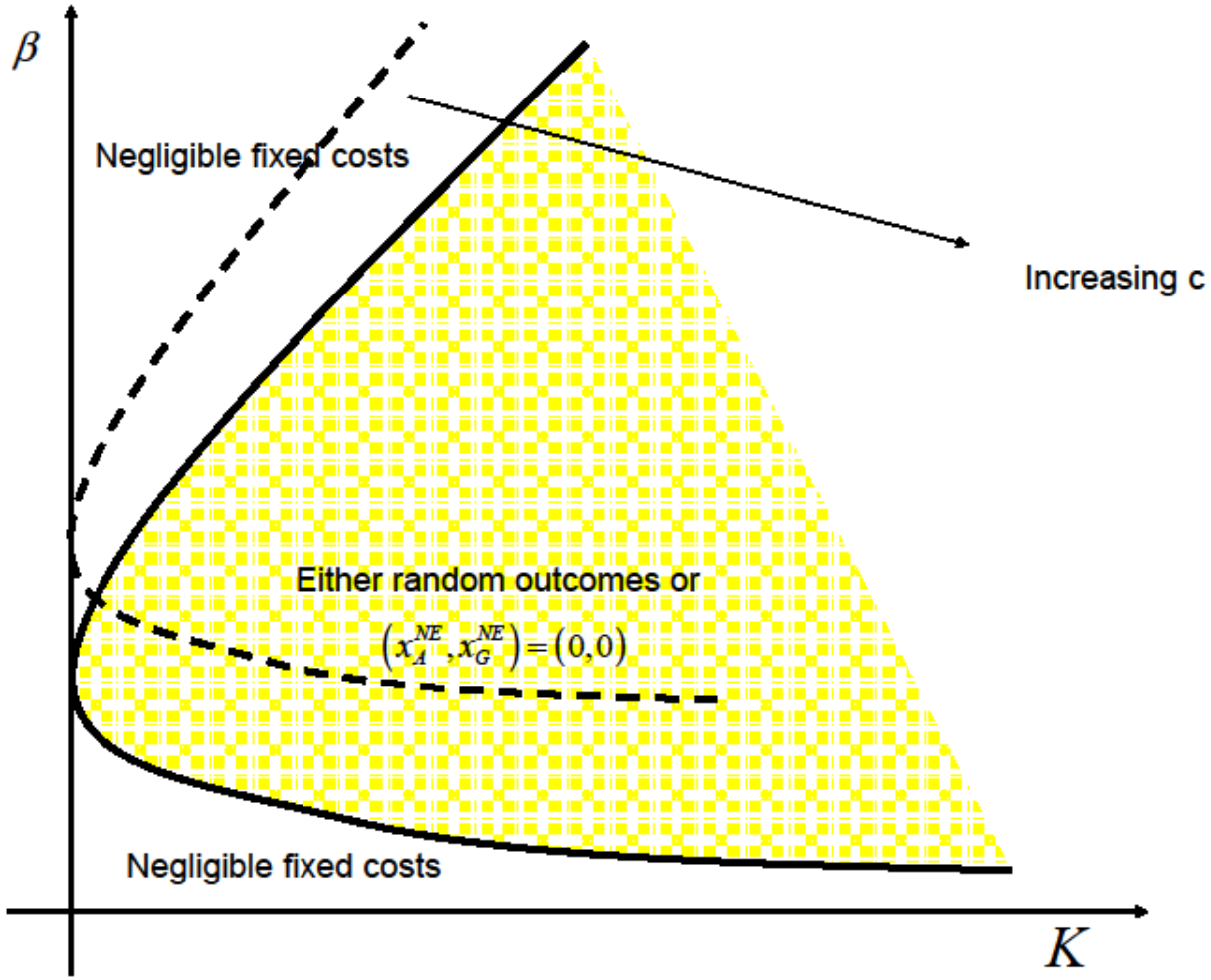
Comparative statics with linear costs & fixed costs

Proposition

When conflict technologies and costs functions are linear without externalities and there are fixed costs, then

1. *an increase in the marginal productivity of effort β has a non monotonic effect on the region of negligible fixed costs;*
2. *an increase in the marginal cost of conflict c has a non monotonic effect on the region of negligible fixed costs;*
3. *an increase in polarization θ has a non monotonic effect on the region of negligible fixed costs.*

A Global View of the Equilibria with linear costs & fixed costs

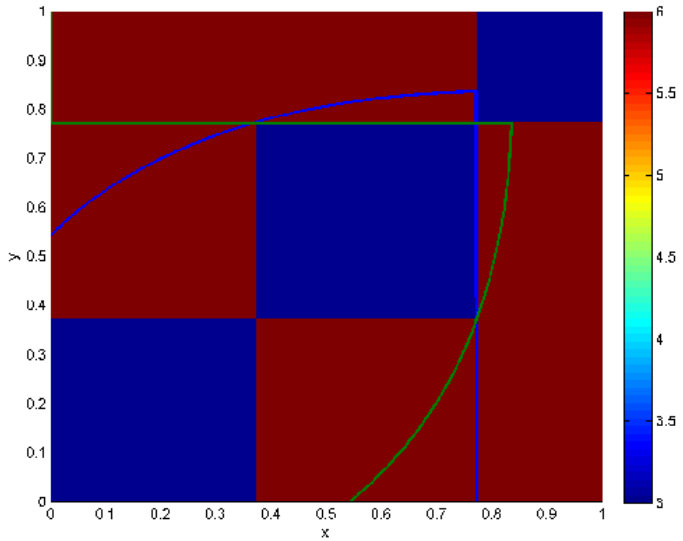


Evolution of conflict with linear costs & fixed costs

Proposition

When conflict technologies and costs functions are linear without externalities and there are fixed costs, then all the possible pure strategy equilibria are fully stable wrt to the best reply dynamics, independently of K, β, c, θ.
However, in the case of non existence of pure strategy equilibria there exists synchronous cycles of period three and asynchronous cycles of period six.

Basins of attractions and cycles with linear costs, fixed costs and no pure strategy equilibria



CASE 2: Conflict technology With Externality

Hypothesis

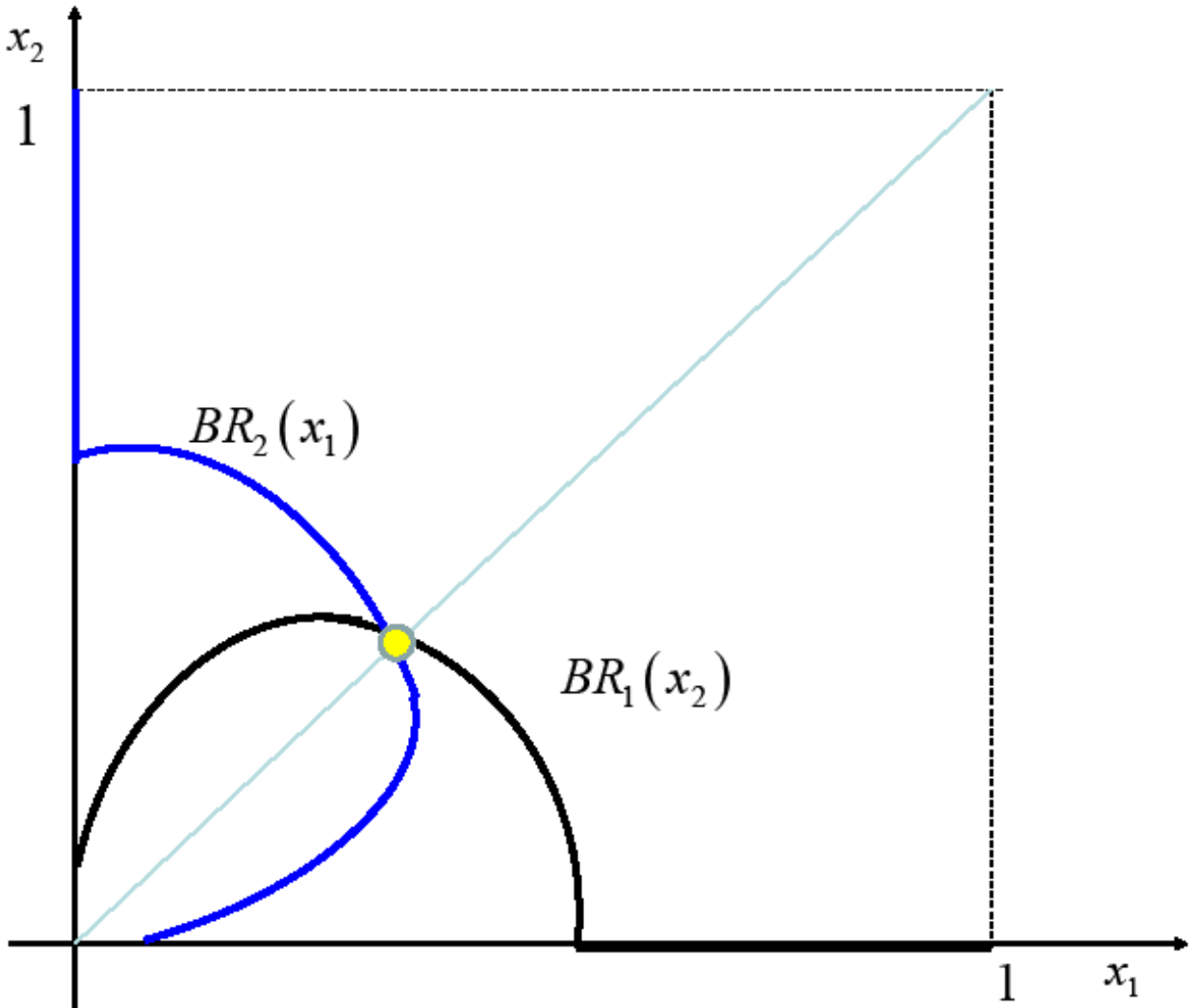
1. *Each agent's effectivity function is*

$$S_i(x_i, x_j) = \beta x_i (1 - \alpha x_j) + 1,$$

where $\alpha \in [0, 1]$ *is a measure of the capacity of the antagonist's in reducing the effectivity of the other agent's effort;*

2. *Each agent's cost function is linear* $c_{i1} = 0, c_{i2} = c, c_{i3} \in \{0, K\}$,
3. *Polarization is constant* $\theta(x_i, x_j) = \theta > 0$.

The Best Replies in case 2 with zero fixed costs



The Equilibrium in case 2 with zero fixed costs

Proposition

When conflict technologies and costs functions are linear with externalities and there are no fixed costs, then there exists a unique symmetric pure strategy Nash equilibrium such that

$$\left(x_1^{NE}, x_2^{NE}\right) = \begin{cases} (0,0) & \frac{\beta}{4c} \in \left[0, \frac{1}{\theta}\right], \alpha \in \left[0, \\ (x^*, x^*) \in (0,1) \times (0,1) & \textit{otherwise} \\ (1,1) & \alpha \in \left[\frac{1}{2}, 1\right] \end{cases}$$

This proposition shows that from a qualitative point of view, the externalities on the conflict technology matters only if the externality parameter is big enough, i.e. $\alpha \in \left[\frac{1}{2}, 1\right]$.

Comparative statics in case 2 with zero fixed costs

Proposition

When conflict technologies and costs functions are linear with externalities and there are no fixed costs, then

1. *an increment in the marginal cost of conflicts c reduces the intensity of conflict in equilibrium;*
2. *an increase of the marginal productivity of the effort β on its effectivity increases the intensity of conflict in equilibrium;*
3. *an increase in polarization θ increases the intensity of conflict in equilibrium;*
4. *an increase in α, increases the likelihood of getting an equilibrium with maximum conflict intensity.*

Evolution of conflict in case 2 with zero fixed costs

Proposition

When conflict technologies and costs functions are linear with externalities and there are no fixed costs, then the possible equilibria are fully stable with respect to the best reply dynamics, independently of α, β, c, θ.

The Equilibrium in case 2 with fixed costs

Proposition

When conflict technologies and costs functions are linear with externalities and there are fixed costs, then the set of pure strategy Nash equilibria might be empty; when it is not empty then:

$$\left(x_1^{NE}(K), x_2^{NE}(K)\right) = \begin{cases} (0,0) & K \text{ big enough} \\ \left(x_1^{NE}, x_2^{NE}\right) & K \text{ intermediate} \\ \left(x_1^{NE}, x_2^{NE}\right) \ \& \ (1,1) & K \text{ negative} \\ & \forall x_j \geq \frac{1}{2\alpha} \end{cases}$$

Hence, the evolution of conflict in case 2-B with negative fixed costs is interesting.

Evolution of conflict in case 2 with fixed costs

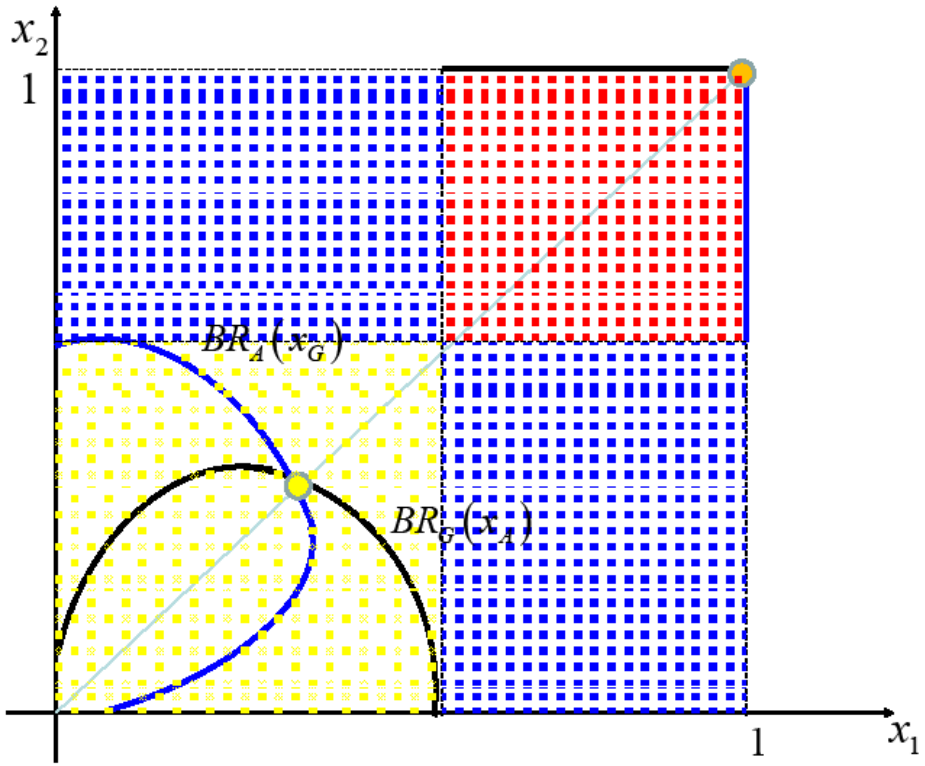
Proposition

hen conflict technologies and costs functions are linear with externalities and there are fixed costs,

1. *if the fixed costs are positive, then all the possible pure strategy equilibria are fully stable, independently of* K, β, c, θ
2. *if fixed cost are negative* $\forall x_j \geq \frac{1}{2\alpha}$*, then the pure strategy equilibria are locally stable and there is the possibility of a two period cycle, where the players alternate combining low with high conflict.*

Best replies and basin of attraction in case 2 with fixed negative costs

In case 2 we have multiple equilibria that can be Pareto ranked and cycles alternating asymmetric intensity of conflict. The following picture reports this case.



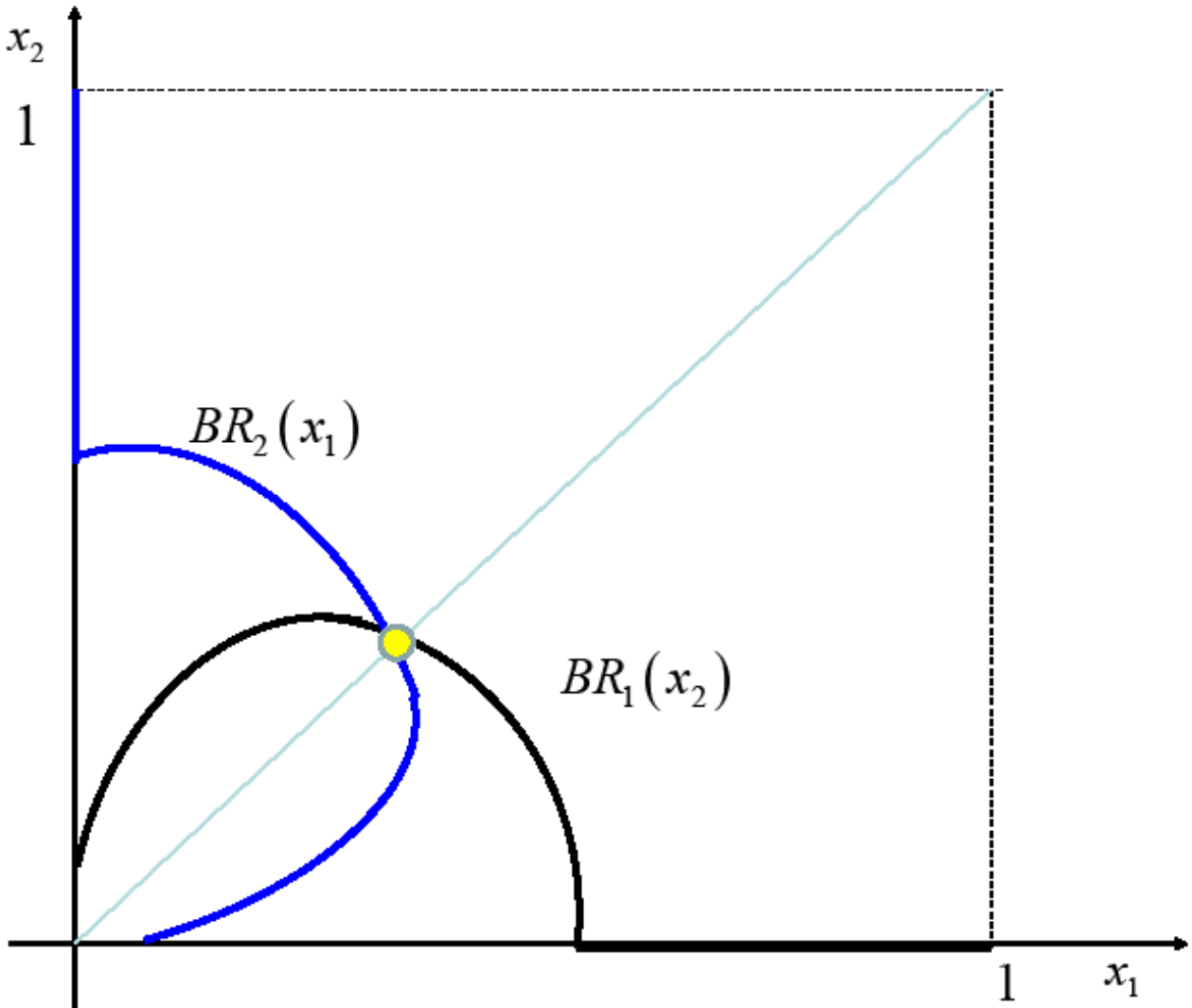
CASE 3: Polarization With Externality

Hypothesis

1. *Each agent's effectivity function is linear* $S(x_i, x_j) = \beta x_i + 1$,
2. *Each agent's cost function is linear* $c_1 = 0, c_2 = c, c_3 \in \{0, K\}$.
3. *Polarization is increasing in both agents' efforts*

$$\theta(x_i, x_j) = \theta + \delta x_i + \delta x_j.$$

The Best Replies in case 2 with zero fixed costs



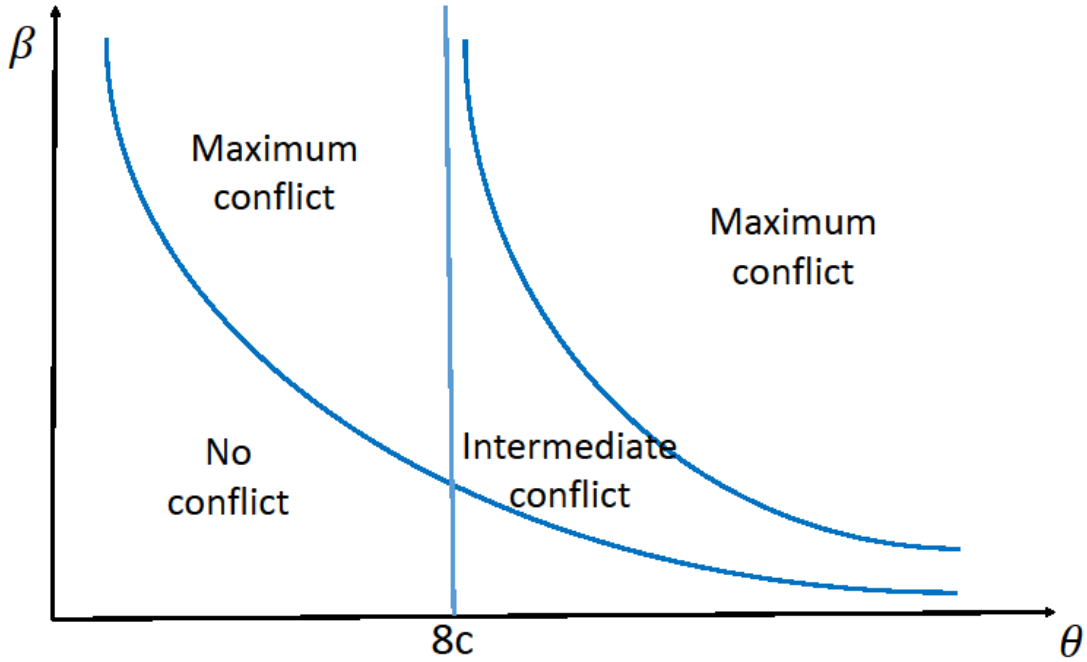
The Equilibrium in case 3 with zero fixed costs

Proposition

When conflict technologies and costs are linear without externalities, there are no fixed costs, and there are externalities in polarizations, then there exists a unique symmetric pure strategy Nash equilibrium such that

$$x_i^{NE} = \begin{cases} 0 & \text{if } \beta \in \left[0, \frac{32c^2+16\delta c+2\delta}{(8+\delta)\theta}\right] \\ \frac{(\beta\theta-2\delta)-8c+\sqrt{(\beta\theta-2\delta)^2-16\delta c}}{8\beta c} & \text{if } \theta \geq 8c \ \& \\ & \beta \in \left[\frac{32c^2+16\delta c+2\delta}{(8+\delta)\theta}; \frac{2\delta+8c}{\theta-8c}\right] \\ 1 & \text{if } \theta \leq 8c \text{ or} \\ & \beta \in \left[\frac{2\delta+8c}{\theta-8c}; \infty\right) \end{cases}$$

A Global View of the Equilibria in case 3 & zero fixed costs



The Limits of the Model

- Two players
- Specific functional forms for
  1. utility
  2. conflict technology
  3. costs functions

The Interpretation of Previous Results - 1

1. Existence, uniqueness and stability are robust to the specification of the conflict technology and of the costs functions without fixed costs

hence in this case

- the interpretation of the stylized facts relies on comparative statics i.e. on the transition from an equilibrium to another as changes in
  - marginal productivity of effort
  - marginal cost of effort
  - **polarization**
- interpretation of stylized facts as a consequence of **structural changes.**

The Interpretation of Previous Results - 2

2. Non existence, multiplicity and interesting dynamics requires the specification of peculiar fixed costs functions

hence

- the interpretation of the stylized facts as dynamic transitions from a region to another relies on the role of costs functions even with slightly complex conflict technologies.

Conclusion

- Our results are partially disappointing since we would prefer to interpret the stylized facts in terms of dynamic transitions
- however
- the model stress the relevance of costs functions in conflict models
- and
- provides a general framework to discuss stylized facts.
- Our future work will focus on asymmetries.