GAME THEORY

STRATEGIC-FORM GAMES

- **MODEL**

DEFINITION: A GAME IS COMPOSED OF TWO ELEMENTS:

- GAME MECHANISM DEFINING THE RULES OF THE GAME
- STRATEGIES DEFINING THE BEHAVIOR OF THE PLAYERS

DEFINITION: GAME MECHANISM IN STRATEGIC FORM (N, A, X, f, u)

- $N = \{1, 2, \dots n\}$ SET OF PLAYERS i
- $A = \{A_1, \dots A_n\}$ A_i SET OF ACTIONS OF i, a_i generic action of i
- $X = \{x_1 \dots x_n\}$ SET OF OUTCOMES x_J
- $f: A_1 \times \dots A_n \longrightarrow X$ OUTCOME FUNCTION (THE LARGER THE BETTER)
- $u = (u_1, \dots u_n)$ $u_i : X \longrightarrow \mathbb{R}$ UTILITY FUNCTION OF i

EXAMPLE: ROCK-PAPER-SCISSORS GAME

$N = \{1, 2\}$

$A_1 = \{R, P, S\} = A_2$

$X = \{W1, W2, T\}$

$W1 \rightarrow$ 1 WINS / 2 LOSES

$W2 \rightarrow$ 2 WINS / 1 LOSES

$T \rightarrow$ TIED

f:

		PLAYER 2 R	P	S
player 1	R	T	W2	W1
	P	W1	T	W2
	S	W2	W1	T

$$u_i(x) = \begin{cases} 1 & x = W_i \\ -1 & x = W_{-i} \\ 0 & x = T \end{cases}$$

EXAMPLE: USUALLY X and f are OMITTED

(N, A, w)

THE TUPLE IS

PLAYER 2

		R	P	S
PLAYER 1	R	0,0	-1,1	1,-1
	P	1,-1	0,0	-1,1
	S	-1,1	1,-1	0,0

EXAMPLE. BACH OR STRAVINSKY GAME (2x2 BATTLE OF SEXES)

A BOY, A GIRL

ATTENDING BACH OR STRAVINSKY?

ATTENDING A CONCERT TOGETHER IS BETTER THAN ALONE BUT EACH PLAYER HAS DIFFERENT PREFERENCES

BOY PREFERS BACH

GIRL PREFERS STRAVINSKY

PLAYER 2 (GIRL)

		B	S
PLAYER 1 (BOY)	B	2,1	0,0
	S	0,0	1,2

(COORDINATION GAME)

EXAMPLE: PRISONERS' DILEMMA

2 PRISONERS

IF BOTH CONFESS → 2 YEARS IN PRISON

IF ONE CONFESSES → 0 YEARS TO THE CONFESSING PRISONER, 3 YEARS TO THE OTHER

IF BOTH DO NOT CONFESS → 1 YEAR IN PRISON

PLAYER 2

		C	NC
PLAYER 1	C	-2,-2	0,-3
	NC	-3,0	-1,-1

OBSERVATION: IN A STRATEGIC-FORM GAME PLAYERS PLAY SIMULTANEOUSLY. THIS IS NOT THE GENERAL CASE. IN GENERAL THE GAME IS CARRIED OUT ON A TREE.

DEFINITION: STRATEGY OF A PLAYER i is a probability distribution over A_i

$$\sigma_i : A_i \to \Delta_i$$ WHERE Δ_i IS THE SET OF probability DISTRIBUTIONS OVER A_i

EXAMPLE:

$$\sigma_i = \begin{cases} 0.2 & R \\ 0.3 & P \\ 0.5 & S \end{cases}$$ Rock Paper Scissors

$$\sigma_i = \begin{cases} 0.4 & B \\ 0.6 & S \end{cases}$$ BACH OR STRAVINSKY

DEFINITION: STRATEGY σ_i IS PURE IF ONLY ONE ACTION IS PLAYED WITH STRICTLY POSITIVE PROBABILITY. OTHERWISE IS MIXED.

EXAMPLE:

$$\sigma_i = \begin{cases} 1 & R \\ 0 & P \\ 0 & S \end{cases}$$ Rock Paper Scissors

DEFINITION: STRATEGY PROFILE $\sigma = (\sigma_1, \dots \sigma_n)$ SPECIFIES ONE STRATEGY PER Agent

EXAMPLE:

$$\sigma_1 = \begin{cases} 1 & R \\ 0 & P \\ 0 & S \end{cases} \quad \sigma_2 = \begin{cases} 1 & R \\ 0 & P \\ 0 & S \end{cases} \quad \sigma = (\sigma_1, \sigma_2)$$

OBSERVATION: WITH MIXED STRATEGIES AGENTS RECEIVE EXPECTED UTILITY (IN EXPECTATION W.R.T. THE STRATEGIES)

EXAMPLE:

$$\sigma_1 = \begin{cases} 0.5 & R \\ 0.5 & P \\ 0 & S \end{cases} \quad \sigma_2 = \begin{cases} 1 & R \\ 0 & P \\ 0 & S \end{cases} \quad \begin{aligned} &\mathbb{E}[u_1(\sigma_1, \sigma_2)] = 0.5 \\ &\mathbb{E}[u_2(\sigma_1, \sigma_2)] = -0.5 \end{aligned}$$

• SOLUTION CONCEPTS

QUESTION: WHAT IS A SOLUTION OF A GAME?

DEFINITION: A SOLUTION CONCEPT DEFINES A SET OF CONDITIONS OVER THE STRATEGIES OF THE AGENTS

DEFINITION: A SOLUTION IS A STRATEGY PROFILE THAT SATISFIES THE CONDITIONS OF A GIVEN SOLUTION CONCEPT

DEFINITION: DOMINANT STRATEGY EQUILIBRIUM $(G_1, \dots, G_n)$ IS SUCH THAT G_i IS PURE AND IT IS THE BEST ACTION AGENT i CAN PLAY FOR ALL THE OPPONENTS' STRATEGIES

EXAMPLE: PRISONER DILEMMA

2

1	C	NC
C	-2, -2	0, -3
NC	-3, 0	-1, -1

DSE (C, C)

TAKING C IS STRICTLY BETTER THAN TAKING NC FOR ALL THE ACTIONS TAKEN BY THE OPPONENTS

DEFINITION: AN ACTION IS STRICTLY DOMINATED IF THERE EXISTS ANOTHER ACTION THAT IS ALWAYS (FOR ALL THE OPPONENTS ACTIONS) BETTER STRICTLY

EXAMPLE: PRISONER'S DILEMMA

NC IS ALWAYS WORSE

2

1	C	NC
C	-2	0
NC	-3	-1

PLAYER 1

2

1	C	NC
C	-2	-3
NC	0	-1

PLAYER 2

EXAMPLE: IN ROCK PAPER-SCISSORS GAME AND IN BACH OR STRAVINSKY NO ACTION IS STRICTLY DOMINATED

DEFINITION: AN ACTION IS WEAKLY DOMINATED IF THERE EXISTS ANOTHER ACTION THAT IS ALWAYS (FOR ALL THE OPPONENTS' ACTIONS) BETTER

EXAMPLE:

2

1	C	D
A	2, 1	0, 2
B	2, 3	3, 1

A IS WEAKLY DOMINATED

OBSERVATION: IF DOMINATED ACTIONS ARE REMOVED AND ONLY ONE ACTION PER PLAYER SURVIVES, THEN A DSE IS FOUND

EXAMPLE:

2

1	E	F	G	H	I
A	2, 2	0, 1	2, 0	0, 2	0, 1
B	3, 3	1, 1	2, 0	1, 0	2, 0
C	2, 2	1, 2	0, 2	1, 0	0, 2
D	0, 1	0, 0	1, 0	1, 1	0, 0

A IS DOMINATED (WEAKLY)
C IS DOMINATED (WEAKLY)
D IS DOMINATED (WEAKLY)
F IS DOMINATED (WEAKLY)
G IS DOMINATED (WEAKLY)
H IS DOMINATED (WEAKLY)
I IS DOMINATED (WEAKLY)

B SURVIVES

E SURVIVES

(B, E) DSE

[redacted]: FURTHER TOPICS NON DISCUSSED DURING THE LECTURES)

- ITERATED DOMINANCE
- DOMINANCE IN MIXED STRATEGIES

OBSERVATION: THERE ARE GAMES WITHOUT DSE
BACH OR STRAVINSKY
ROCK - PAPER - SCISSORS

DEFINITION: BEST RESPONSE OF PLAYER i GIVEN THE STRATEGIES OF THE OPPONENTS $-i$: SET OF ACTIONS THAT MAXIMIZE THE UTILITY OF PLAYER i GIVEN THE STRATEGIES OF THE OPPONENTS

EXAMPLE: $BR_1(\sigma_2)$ IN PRISONER DILEMMA

$$= \{ C \text{ FOR ALL } \sigma_2$$

EXAMPLE: $BR_1(\sigma_2)$ IN BACH OR STRAVINSKY

$$= \begin{cases} B & \sigma_2 = \begin{cases} \geq 1/3 & B \\ \leq 2/3 & S \end{cases} \\ S & \sigma_2 = \begin{cases} \leq 1/3 & B \\ \geq 2/3 & S \end{cases} \end{cases}$$

IF $\sigma_2 = \begin{cases} 1/3 & B \\ 2/3 & S \end{cases}$ THEN ANY σ_1 IS A BEST RESPONSE

DEFINITION: NASH equilibrium IS $(\sigma_1, \sigma_2, \dots \sigma_n)$ SUCH THAT $\sigma_i \in BR_i(\sigma_{-i}) \ \forall i$

EXAMPLE: BACH OR STRAVINSKY GAME

(B, B) IS A NASH Equilibrium

(S, S) IS A NASH equilibrium

$(2/3 \oplus 1/3,\ 1/3 \oplus 2/3)$ IS A NASH Equilibrium

PROPERTY: EVERY FINITE GAME ADMITS AT LEAST ONE NASH Equilibrium IN MIXED STRATEGIES

ROPERTY: • A DSE IS ALSO A NASH equilibrium
• THE REVERSE DOES NOT HOLD

LGORITHM: TO FIND A PURE STRATEGY equilibrium, BEST RESPONSE DYNAMICS CAN BE USED

4, 5	3, 7	2, 8	9, 3	5, 4	7, 7
3, 1	0, 5	5, 2	4, 4	3, 1	2, 2
6, 4	0, 0	3, 1	5, 6	4, 3	0, 0
3, 0	3, 3	0, 0	7, 7	2, 1	8, 8
0, 1	4, 0	2, 2	5, 3	0, 0	2, 4

BR2 BR1 BR2 BR1 BR1 BR2

1. TURNS
2. AT EACH TURN A PLAYER PLAYS ITS BEST RESPONSE
3. IF A STABLE STATE IS ACHIEVED, IT IS A NASH equilibrium

THE INITIAL PROFILE IS CHOSEN AT RANDOM

ROPERTY: BEST RESPONSE DYNAMICS ALGORITHM MAY NOT CONVERGE EVEN IF THERE IS A PURE-STRATEGY NASH equilibrium (USUALLY A TABU LIST IS USED)

URTHER TOPICS (NON DISCUSSED DURING THE LECTURES):
- HOW TO COMPUTE A NASH equilibrium in MIXED STRATEGIES
- WHAT IS THE COMPLEXITY OF FINDING A NASH equilibrium (IN PURE AND IN MIXED STRATEGIES)

BSERVATION: NASH equilibrium requires all the PLAYERS KNOW THE PAYOFFS (UTILITIES) IF THE PLAYERS START PLAYING A NASH

IF AGENTS REACH A NASH BY BEST RESPONSE DYNAMICS NON REQUIREMENT OVER THE KNOWLEDGE OF THE AGENTS IS NEEDED

FURTHER TOPICS (NON DISCUSSED DURING THE LECTURES) :

- CONGESTION GAMES AND DYNAMICS
- BAYESIAN GAMES
- EXTENSIVE-FORM GAMES
- NASH equilibrium REFINEMENT
- EVOLUTIONARY GAME THEORY
- ZERO SUM GAMES, MAX MIN / MIN MAX

DEFINITION: A BAYESIAN GAME IS A GAME IN WHICH EACH AGENT CAN BE OF DIFFERENT TYPES, EACH WITH A GIVEN PROBABILITY

DEFINITION: A TYPE $\theta_{i,T}$ SPECIFIES THE PAYOFFS OF AN AGENT

EXAMPLE:

1 \ 2	B	S
B	2,1	0,0
S	0,0	1,2

$\theta_{2.1}$

$\omega_{2.1} = 0.3$

1 \ 2	B	S
B	2,0	0,2
S	0,1	1,0

$\theta_{2.2}$

$\omega_{2.2} = 0.7$

OBSERVATION: IN GENERAL, IN A BAYESIAN GAME, EACH PLAYER i IS CHARACTERIZED BY A SPACE OF TYPES $\Theta_i := \{\theta_{i,1}, \dots \theta_{i,K}\}$

DEFINITION: THE NORMAL FORM OF A BAYESIAN GAME IS A STRATEGIC FORM GAME IN WHICH: 1) THE ACTIONS ARE TUPLES CONTAINING ONE ACTION (OF THE ORIGINAL GAME) PER TYPE, AND 2) THE PAYOFFS ARE THE EXPECTED UTILITIES FROM THE COMBINATION OF ACTIONS

EXAMPLE:

BB → $\theta_{2,1}$, $\theta_{2,2}$

		2			
		BB	BS	SB	SS
1	B	2, 0.3	0.6, 1.7	1.4, 0	0, 1.4
	S	0, 0.7	0.7, 0	0.3, 1.3	1, 0.6

DEFINITION

A Bayes-Nash equilibrium is a Nash equilibrium of the normal form game of a Bayes Nash game