# ECONOMIC MECHANISMS

## MODEL

**DEFINITION:** A social choice function $f: \Theta_1 \times \dots \times \Theta_n \to X$ assigns an outcome $x \in X$ to each possible combination of types (profile of types).

**EXAMPLE:** $X = \{x, y, z\}$, $N = \{1, 2\}$, $\Theta_1 = \{\theta_1\}$, $\Theta_2 = \{\theta_{2.1}, \theta_{2.2}\}$

$$u_1(x) > u_1(y) > u_1(z)$$

$$u_{2.1}(z) > u_{2.1}(y) > u_{2.1}(x)$$

$$u_{2.2}(y) > u_{2.2}(x) > u_{2.2}(z)$$

Suppose $\begin{cases} f(\theta_1, \theta_{2.1}) = y \\ f(\theta_1, \theta_{2.2}) = x \end{cases}$

IT IS CLEAR THAT $\theta_{2.2}$ WILL NEVER REPORT TRUTHFULLY HER TYPE. INDEED, IF she REPORTS $\theta_{2.1}$, she WILL GENERATE THE OUTCOME $y$ THAT IS BETTER FOR $\theta_{2.2}$

**EXAMPLE:** $X = \{1, \dots, n\}$ $N = \{1, \dots, n\}$ $\Theta_i = [0, 1]$

$$u_i(x) = \begin{cases} \theta_i & x = i \\ 0 & x \neq i \end{cases}$$

Suppose $f(\theta_1, \dots, \theta_n) = \arg\max_{i \in N} \{\theta_i\}$

THE WINNER IS THE AGENT WITH THE LARGEST $\theta_i$

IF MULTIPLE AGENTS HAVE THE SAME $\theta_i$, THE WINNER IS SELECTED RANDOMLY

IT IS CLEAR THAT ALL THE AGENTS DO NOT REPORT TRUTHFULLY THEIR TYPE. INDEED, EVERY AGENT WILL REPORT $\hat{\theta}_i = 1$ ( $\hat{\theta}_i$ DENOTES THE TYPE REPORTED BY $i$)

**EXAMPLE:** $X = \{(i, p_1, \dots, p_n) : i \in N, \ p_i \in \mathbb{R}\}$  $N = \{1, \dots, n\}$

$\Theta_i = [0,1]$

$\boxed{(i, p_1, \dots, p_n) \to i \text{ IS THE WINNER, } p_J \text{ IS THE PAYMENT OF } J}$

$$w_i(J, p_1, \dots, p_n) = \begin{cases} -p_J + \theta_J & \text{IF } J = i \\ -p_i & \text{OTHERWISE} \end{cases}$$

SUPPOSE $f \to (i^*, p_1, \dots, p_n)$ SUCH THAT:

$$i^* = \arg\max_{J \in N} \{\theta_J\}$$

$$p_k = \begin{cases} 0 & \text{IF } k \neq i^* \\ \theta_k & \text{IF } k = i^* \end{cases}$$

(FIRST PRICE AUCTION: THE WINNER PAYS HER BID)

IT IS CLEAR THAT THE AGENTS DO NOT REPORT TRUTHFULLY AS SHOWN BY THE FOLLOWING COUNTEREXAMPLE

$\theta_1 = 2 \qquad \theta_2 = 3 \qquad \theta_3 = 1$

THE OPTIMAL REPORTED TYPES ARE:

$\hat{\theta}_1 = 2 \qquad \hat{\theta}_2 = 2 + \varepsilon \qquad \hat{\theta}_3 = 1$ WITH $\varepsilon \to 0^+$, but $\varepsilon \neq 0$

**EXAMPLE:** AS THE ABOVE, EXCEPT:

$$p_k = \begin{cases} 0 & \text{IF } k \neq i^* \\ \max_{J \in N/k} \{\theta_J\} & \text{IF } k = i^* \end{cases}$$

(SECOND PRICE AUCTION: THE WINNER PAYS THE SECOND HIGHEST BID)

REPORTING TYPES TRUTHFULLY IS AN OPTIMAL STRATEGY (IT IS NOT UNIQUE)

**DEFINITION** AN ~~ECONOMIC~~ MECHANISM (FROM HERE ON ONLY MECHANISM) IS $(A_1, \dots, A, f)$ WHERE $f$ is an OUTCOME function

**OBSERVATION:** AN (ECONOMIC) MECHANISM COMBINED WITH TYPES, PROBABILITY DISTRIBUTIONS OVER THE TYPES, AND UTILITY FUNCTIONS IS A BAYESIAN GAME.

- IMPLEMENTATION

**DEFINITION:** A MECHANISM $(A_1, \dots, A_n, g)$ IMPLEMENTS SOCIAL CHOICE function $f$ IF THERE IS AN equilibrium STRATEGY PROFILE $(\sigma_1^*, \dots \sigma_n^*)$ OF THE ~~Bay~~ GAME INDUCED BY THE MECHANISM SUCH THAT:

$$g(\sigma_1^*(\theta_1), \dots \sigma_n^*(\theta_n)) = f(\theta_1, \dots \theta_n) \text{ for every}$$

$$(\theta_1, \dots \theta_n) \in \Theta_1 \times \dots \times \Theta_n.$$

WHERE $\sigma_i^*(\theta_i)$ IS THE OPTIMAL STRATEGY OF $i$ WHEN THE TYPE IS $\theta_i$

**OBSERVATION:** IMPLEMENTATION MEANS THAT THE EQUILIBRIUM OUTCOME OF MECHANISM IS THE OUTCOME OF THE SOCIAL CHOICE FUNCTION FOR EVERY TYPE PROFILE

**OBSERVATION:** IN THE CASE THERE ARE MULTIPLE EQUILIBRIA, WE JUST REQUIRE THAT THERE IS ONE EQUILIBRIA OUTCOME THAT IS THE OUTCOME OF THE SOCIAL CHOICE FUNCTION

**DEFINITION:** A DIRECT REVELATION MECHANISM IS A MECHANISM IN WHICH $A_i = \Theta_i$ AND $g(\Theta) = f(\Theta)$

**DEFINITION:** SOCIAL CHOICE FUNCTION $f$ IS TRUTHFULLY IMPLEMENTABLE IF THE DIRECT REVELATION MECHANISM $(\Theta_1, \dots \Theta_n, f)$ has an equilibrium $(\sigma_1^*, \dots \sigma_n^*)$ SUCH THAT $\sigma_i^*(\theta_i) = \theta_i$ FOR ALL $i$

**EXAMPLE:** FIRST PRICE AUCTION IS NOT TRUTHFULLY IMPLEMENTABLE

SECOND PRICE AUCTION IS TRUTHFULLY IMPLEMENTABLE

[redacted]

**OBSERVATION:** THE DEFINITION OF TRUTHFULLY IMPLEMENTABLE DOES NOT SPECIFY THE SOLUTION CONCEPT.
MULTIPLE FORMS OF TRUTHFULLY IMPLEMENTABILITY ARE POSSIBLE. FOR INSTANCE:

- DSE
- Bayes - NASH

[redacted]

**OBSERVATION:** THE DEFINITIONS OF:

- IMPLEMENTATION OF
- TRUTHFULLY IMPLEMENTABLE

IN DOMINANT STRATEGIES ARE AS THE ORIGINAL DEFINITION EXCEPT THEY REQUIRE THE equilibrium TO BE DSE

**PROPOSITION** [REVELATION PRINCIPLE] SUPPOSE THAT THERE EXISTS A MECHANISM $(A_1, \dots A_n, g)$ THAT IMPLEMENTS $f$ IN DSE, THEN $f$ IS TRUTHFULLY IMPLEMENTABLE IN DSE.

**OBSERVATION:** TO IDENTIFY THE SET OF SOCIAL CHOICE FUNCTIONS THAT ARE IMPLEMENTABLE IN DOMINANT STRATEGIES WE NEED ONLY IDENTIFY THOSE THAT ARE TRUTHFULLY IMPLEMENTABLE.

**PROPOSITION** [GIBBARD & SATTERTHWAITE THEOREM; JUST THE IDEA] ONLY DICTATORIAL $f$ ARE IMPLEMENTABLE IN GENERAL THAT IS $f$ WHOSE OUTCOME DEPENDS ONLY ON A SINGLE AGENT

**DEFINITION:** [QUASI LINEAR ENVIRONMENT] THE UTILITY OF EACH AGENT AND THE OUTCOMES HAVE A SPECIFIC STRUCTURE:

- OUTCOMES $x = (y, p_1, \dots p_n)$ WHERE $y$ IS SAID THE ALLOCATION, WHILE $p_i$ IS A PAYMENT
- UTILITY FUNCTION ARE:

$$u_i(x, \theta_i) = \underbrace{v_i(y, \theta_i)}_{\text{VALUATION OVER THE ALLOCATION}} - \underbrace{p_i}_{\text{PAYMENT}}$$

OBSERVATION: AUCTIONS ARE PROBLEMS DEFINED IN QUASI-LINEAR ENVIRONMENTS

- Vickrey Clarck Groves

PROPOSITION: IF $y^* = \arg\max_{y^*} \left\{ \sum_{i \in N} v_i(y, \theta_i) \right\}$

THAT IS, $y^*$ IS THE ALLOCATION MAXIMIZING THE CUMULATIVE VALUE (aka SOCIAL WELFARE)

THEN DS TRUTHFULNESS IMPLEMENTATION IS POSSIBLE IF:

$$p_i = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(y^*, \theta_j)$$

WHERE $h_i(\theta_{-i})$ IS a function THAT DOES NOT DEPEND ON $\theta_i$

DEFINITION: A DIRECT REVELATION MECHANISM IN WHICH $f$ IS DEFINED AS:

$$y^* = \arg\max_{y} \left\{ \sum_{i \in N} v_i(y, \theta_i) \right\}$$

$$p_i = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(y^*, \theta_j)$$

IS SAID a Groves MECHANISM

PROOF SKETCH: [TRUTHFULNESS OF GROVES MECHANISM]

EACH AGENT TRIES TO MAXIMIZE ITS UTILITY

$$u_i(y, \theta_i) = v_i(y, \theta_i) - h_i(\theta_{-i}) + \sum_{j \neq i} v_j(y, \theta_j) =$$

$$= \sum_j v_j(y, \theta_j) - h_i(\theta_{-i})$$

MAXIMIZING $u_i$ IS POSSIBLE ONLY MAXIMIZING $\sum_j v_j(y, \theta_j)$
W.R.T. $\theta_i$

GIVEN THAT $h_i(\theta_{-i})$ DOES NOT DEPEND ON $\theta_i$

THEREFORE TRUTHFULLY REPORTING IS OPTIMAL

OBSERVATION: NO PAYMENTS IS NOT POSSIBLE WITH GROVES MECHANISMS

$$p_i = 0 = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(y^*, \theta_j) \rightarrow$$

$$h_i(\theta_{-i}) = \sum_{j \neq i} v_j(y^*, \theta_j)$$

BUT $y^*$ DEPENDS ON $\theta_i$

OBSERVATION: REPORTING $\hat{\theta}_i \neq \theta_i$ DOES NOT MAXIMIZE IN GENERAL $\sum_j v_j(y, \hat{\theta}_j)$. THAT IS, THE MECHANISM MAXIMIZES $\sum_j v_j(y, \hat{\theta}_j)$ RETURNING A $y^*$, BUT THIS ALLOCATION IS GENERALLY DIFFERENT FROM THE ALLOCATION MAXIMIZING $\sum_j v_j(y, \theta_j)$

EXAMPLE: $N = \{1, 2, 3\}$ $\quad Y = \{1, 2, 3\}$

$$v_i(y, \theta_i) = \begin{cases} \theta_i & y = i \\ 0 & y \neq i \end{cases}$$

$$\theta_1 = 1 \quad \theta_2 = 2 \quad \theta_3 = 3 \rightarrow y^* = 3 \quad \sum_j v_j(y^*, \theta_j) = 3$$

$$\hat{\theta}_1 = 1 \quad \hat{\theta}_2 = 2 \quad \hat{\theta}_3 = 1.5 \rightarrow y^* = 2 \quad \sum_j v_j(y^*, \theta_j) = 2$$

OBSERVATION: THE EQUILIBRIUM IS WEAKLY DOMINANT STRATEGIES GIVEN THAT THERE ARE $\hat{\theta}_i \neq \theta_i$ SUCH THAT THE OUTCOME IS THE SAME OF THAT WITH $\theta_i$

EXAMPLE: SECOND PRICE AUCTION IS A SPECIAL GROVES MECHANISM IN WHICH $h_i(\theta_{-i})$ IS DEFINED AS FOLLOWS:

$$h_i(\theta_{-i}) = \begin{cases} \text{HIGHEST OFFER } \max_j \{\theta_j\} & \text{IF } i \neq i^* \\ \text{SECOND HIGHEST OFFER} & \text{IF } i = i^* \end{cases}$$

THIS AUCTION IS KNOWN ALSO AS VICKREY AUCTION

EXAMPLE: FIRST PRICE AUCTION IS NOT A GROVES MECHANISM GIVEN THAT $p_i$ DEPENDS ON $\theta_i$

PROPERTY: WHEN $v_i(y, \theta_i)$ CAN BE ANY, ONLY GROVES MECHANISMS IMPLEMENT TRUTHFULLY IN DOMINANT STRATEGIES SOCIAL CHOICE FUNCTIONS

DEFINITION: [CLARKE PIVOT] $h_i(\theta_{-i}) = \max_y \sum_{j \neq i} v_j(y, \theta_j)$

OBSERVATION: VICKREY AUCTION IS A SPECIAL CASE OF GROVES MECHANISMS WITH CLARKE PIVOT

PROPERTY: WITH CLARKE PIVOT, $p_i \geq 0$ IN EVERY SITUATION (THE PROOF IS UPTO YOU)

PROPERTY: A MECHANISM IS WEAKLY BUDGET BALANCED IF THE MECHANISM HAS NO DEFICIT, THAT IS:

$$\sum p_i \geq 0 \quad \text{IN EVERY SITUATION}$$

OBSERVATION: GROVES MECHANISM WITH CLARKE PIVOT IS WEAKLY BUDGET BALANCED

DEFINITION: A MECHANISM IS BUDGET BALANCED (STRICTLY) IF

$$\sum p_i = 0$$

DEFINITION: A MECHANISM IS ALLOCATIVELY EFFICIENT IF

$$y^* \in \arg\max_y \left\{ \sum_j v_j(y, \theta_j) \right\}$$

**DEFINITION:** A MECHANISM IS INDIVIDUALLY RATIONAL IF AN AGENT HAS ALWAY POSITIVE UTILITY

$w_i(Y, \theta_i) \geq 0$ IN EVERY SITUATION UNDER TRUTHFULL REPORTING

**PROPERTY:** VCG IS INDIVIDUALLY RATIONAL

**OBSERVATION:** A SUMMARY OF THE PROPERTIES OF THE ECONOMIC MECHANISM

TRUTHFULL $\longrightarrow$ ECONOMIC STABILITY

INDIVIDUALLY RATIONAL $\longrightarrow$ EVERY AGENT PREFERS TO TAKE PART TO THE MECHANISM RATHER THAN ABSTEINING

WEAKLY BUDGET BALANCED $\longrightarrow$ THE MECHANISM IS NOT IN DEFICIT (IT CAN GENERATE PROFIT)

STRICTLY BUDGET BALANCED $\longrightarrow$ THE MECHANISM HAS NO PROFIT

ALLOCATIVELY EFFICIENT $\longrightarrow$ THE SOCIAL WELFARE (INCLUDED THE MECHANISM) IS MAXIMIZED

- LINEAR ENVIRONMENT

**DEFINITION:** LINEAR ENVIRONMENT IS A SPECIFIC CASE OF QUASI-LINEAR ENVIRONMENT IN WHICH:

- $$v_i(Y, \theta_i) = \underbrace{\theta_i \cdot \psi_i(Y)}$$

FACTORIZED: $\psi_i$ DOES NOT DEPEND ON $\theta_i$, BUT ONLY ON $Y$

**EXAMPLE:** THE VICKREY AUCTION OPERATES IN LINEAR ENVIRONMENT

- $$v_i(Y, \theta_i) = \theta_i \cdot \psi_i(Y)$$

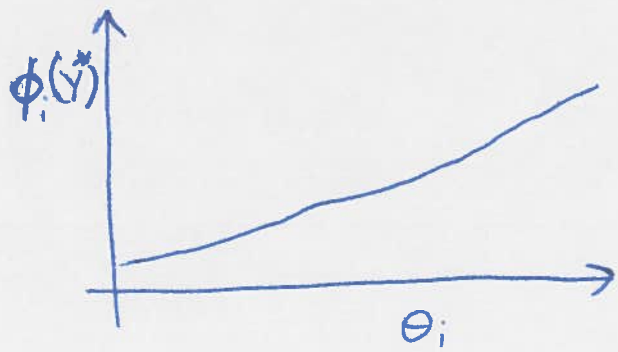
$$\psi_i(Y) = \begin{cases} 0 & Y \neq i \\ 1 & Y = i \end{cases}$$

**DEFINITION:** AN ALLOCATION FUNCTION IS SAID MONOTONIC IN LINEAR ENVIRONMENT IF:

$$\phi_i(Y^*) \text{ IS MONOTONICALLY INCREASING IN } \theta_i$$

THAT IS: THE LARGER $\theta_i$, THE LARGER OF THE VALUATION $\phi_i$ OF THE ALLOCATION $Y^*$

**OBSERVATION:** IN PRACTICE, FOR EVERY $\theta_{-i}$ WE HAVE




**PROPERTY:** ONLY SOCIAL CHOICE FUNCTION WITH MONOTONIC $Y^*$ ARE TRUTHFULLY IMPLEMENTABLE IN DOMINANT STRATEGIES

**PROPERTY:** EVERY SOCIAL CHOICE FUNCTION WITH MONOTONIC $Y^*$ IS TRUTHFULLY IMPLEMENTABLE IN DOMINANT STRATEGIES

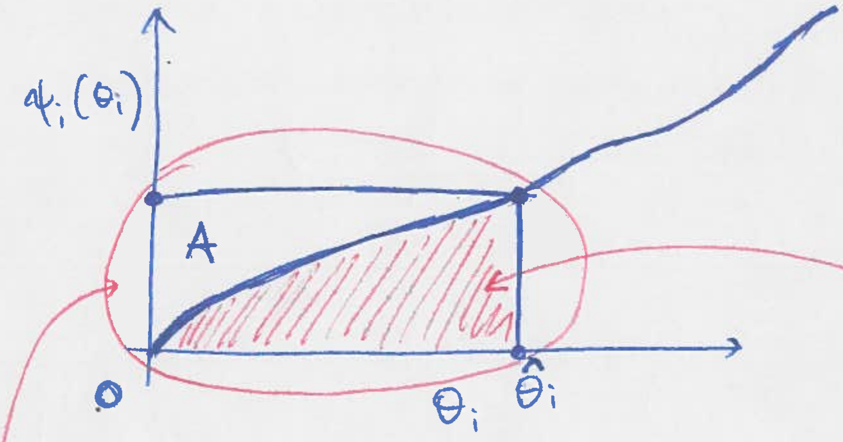
**PROPERTY:** [MYERSON] IN ORDER TO HAVE TRUTHFULNESS IN DOMINANT STRATEGIES PAYMENTS MUST BE OF THE FORM

$$p_i(\hat{\theta}) = h_i(\hat{\theta}_{-i}) + \hat{\theta}_i \, \phi_i(Y^*(\hat{\theta})) - \int_{\underline{\theta}_i}^{\hat{\theta}_i} \phi_i(Y^*(\theta_i, \hat{\theta}_{-i}))\, d$$

WHERE $\underline{\theta}_i$ IS THE MINIMUM VALUE FOR $\theta_i$

OBSERVATION: WE CAN HAVE MORE MECHANISMS, DIFFERENT FROM GROVES MECHANISMS

EXAMPLE:




$$p_i = h_i(\hat{\theta}_{-i}) + \underbrace{\hat{\theta}_i \cdot \psi_i(Y^*(\dots))}_{\text{THE AREA OF THE RECTANGLE}} - \underbrace{\int_0^{\hat{\theta}_i} \psi_i(Y^*(\dots))\,d\theta_i}_{\text{THE AREA UNDER THE CURVE}}$$

$$\Rightarrow p_i = h_i(\hat{\theta}_{-i}) + \underbrace{A}_{\text{THE AREA BETWEEN THE CURVE AND THE RECTANGLE}}$$

OBSERVATION:

$$u_i = \int_0^{\hat{\theta}_i} \psi_i(Y^*(\dots))\,d\theta_i - h_i(\hat{\theta}_{-i}) \Big|^{\text{UNDER TRUTHFULNESS}}$$

GIVEN THAT $\hat{\theta}_i \psi_i(Y^*(\dots))$ IS THE VALUATION OF THE ALLOCATION AND APPEARS IN THE PAYMENT

## • MAXIMAL IN ITS RANGE

**DEFINITION:** A RANGE $Y'$ IS A SUBSET OF OUTCOMES $Y$, SUCH THAT $Y'$ DOES NOT DEPEND ON THE REPORTS

**DEFINITION:** A MAXIMAL-IN-ITS-RANGE MECHANISM IS A MECHANISM SELECTING THE OUTCOME MAXIMIZING THE SOCIAL WELFARE IN THE RANGE

**PROPERTY:** MAXIMAL-IN-ITS-RANGE MECHANISMS ARE GROVES MECHANISMS

**FURTHER READINGS** (NON DISCUSSED DURING THE LECTURES):

- WEIGHTED GROVES MECHANISMS
- REDISTRIBUTION MECHANISMS
- BAYESIAN IMPLEMENTATION