Double Auctions

Scenario:

- There are n buyers

$$B = \{ b_1 = 11,\ b_2 = 7,\ b_3 = 5,\ b_4 = 3,\ b_5 = 2,\ b_6 = 1 \}$$

- There are n sellers

$$S = \{ s_1 = 5,\ s_2 = 6,\ s_3 = 7,\ s_4 = 8,\ s_5 = 9,\ s_6 = 10 \}$$

- If there are more buyers or more sellers, append sellers with value +∞ or buyers with value 0 to obtain the same number

- s_i and b_T can make a deal if $s_i < b_T$

Bipartite Graph: Sellers, buyers, edges → $w(s_i, b_T) = b_T - s_i$ if $b_T - s_i \geq 0$, 0 otherwise

Matching: Two possible matchings

- Maximum size matching (largest number of matched players)

$$M = \{ (s_1, b_3), (s_3, b_2), (s_4, b_1) \}$$

Social welfare $= (5-5) + (7-7) + (11-8) = 3$

- Maximum weighted matching (maximizing the social welfare)

$$M = \{ (s_1, v_2), (s_2, v_1) \}$$

Social welfare $= (7-5) + (11-6) = 7$

OBSERVATION: IN PRACTICE, THE MAXIMIZATION OF THE SOCIAL WELFARE IS ADOPTED

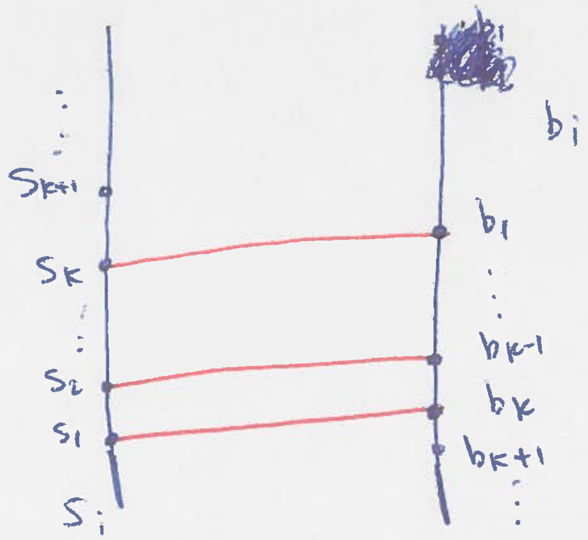
PROBLEM: FINDING THE BEST ALLOCATION MAXIMIZING THE SOCIAL WELFARE

ALGORITHM:

- SORT SELLERS $s_1 \le s_2 \le \dots \; s_n$
- SORT BUYERS $b_1 \ge b_2 \ge \dots \; b_n$
- FIND THE MAXIMUM k SUCH THAT:

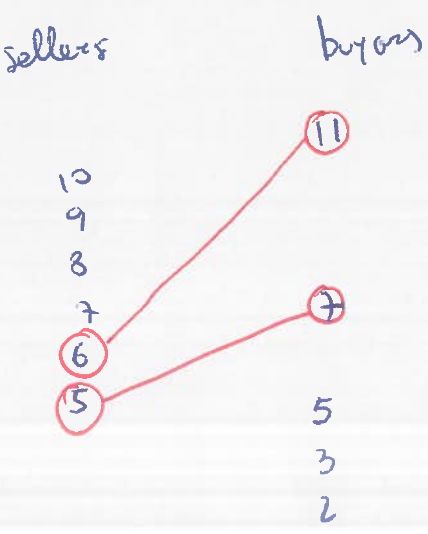
$$M = \{(s_1, b_k), (s_2, b_{k-1}), \dots (s_k, b_1)\} \subseteq \text{FEASIBLE}$$

$$\Rightarrow b_k \ge s_1 \; \dots \; b_1 \ge s_k \Rightarrow s_{k+1} > b_1 \text{ OR}$$
$$s_1 > b_{k+1}$$



- THE ALGORITHM RUNS IN LINEAR TIME

EXAMPLE:



MAXIMUM SOCIAL WELFARE

PAYMENTS [VCG] THE VCG PAYMENT CAN BE WRITTEN AS USUAL

$$SW^*(\theta_{-i}) - SW^*_{-i}(\theta)$$

EXAMPLE:

$$p_{b_1} = (2) - (-4) = 6$$

THE OPTIMAL MATCHING IS (s_1, b_2)

$$\Downarrow$$

$$7 - 5 = 2$$

$$b_2 - (s_1 + s_2) = 7 - (5 + 6) = -4$$

$$p_{b_2} = (6) - (0) = 6$$

$$b_1 - (s_1 + s_2) = 11 - (5 + 6) = 0$$

(s_1, b_1) OPTIMAL MATCHING

$$p_{s_1} = -7$$

$$p_{s_2} = -7$$

$$\Longrightarrow \sum p_i = -2 \Longrightarrow \text{NO WBB}$$

OBSERVATION: WHY IS VCG NOT WBB IN DOUBLE auction SETTINGS?

- CALL M^* the optimal MATCHING, OPT is the VAL
- $V(OPT) = \sum_{M^*} w(e)$
- $P(b_j) = V(OPT_{-b_j}) - [V(OPT) - b_j]$
- $P(s_i) = V(OPT_{-s_i}) - [V(OPT) + s_i]$

- The auctioneer receives $\sum_{S_i \in M^*} p(S_i) + \sum_{b_T \in M^*} p(b_T)$

$\Rightarrow$

$$\sum_{S_i \in M^*} v(OPT_{-S_i}) + \sum_{V_T \in M^*} v(OPT_{-b_T}) - (2|M^*|-1)v(OPT)$$

- $m^* = |M^*|$, $b_{m^*+1} - S_{m^*+1} \leq 0$

- $v(OPT_{-b_J}) \leq v(OPT) - b_J + b_{m^*+1}$

  $v(OPT_{-S_i}) \leq v(OPT) + S_i - S_{m^*+1}$

$\Rightarrow \sum p_i \leq m^*(b_{m^*+1} - S_{m^*+1}) \leq 0$

OBSERVATION: VCG cannot be adopted with Double Auctions

QUESTION: Can we have DS truthfulness, IR and WBB together?

ANSWER: Yes, we can by using Preston McAfee Double Auction Mechanism

MECHANISM: Sort b_i : $b_1 \geq b_2 \geq \dots \geq b_n$

Sort S_J : $S_1 \leq S_2 \leq \dots \leq S_n$

Call k the last pair (b_k, S_k) such that $b_k \geq S_k$

(1) All b_i and S_i with $i < k$ are matched and $p_{b_i} = b_k$ and $p_{S_J} = -S_k$

The mechanism revenue is $(k-1) \cdot (b_k - S_k)$

(2) If $\frac{1}{2}(S_{k+1} - b_{k+1}) \in [S_k, b_k]$ then all the players pay/receive $\frac{1}{2}(\dots)$ and all the first k player are matched

① CAN SOME AGENT GAIN MORE BY MISREPORTING ?

$\hat{b}_1 = b_1 + \varepsilon \Rightarrow P_{b_1} = 8$ FOR EVERY $\varepsilon > 0$

$\hat{b}_1 = b_1 - \varepsilon \Rightarrow P_{b_1} = 8$ FOR $\varepsilon \geq 4 \Rightarrow w_{b_1} = 4$

$\varepsilon > 4 \Rightarrow b_1$ IS NOT MATCHED $\Rightarrow w_{b_1} = 0$

$\Rightarrow b_1$ DOES NOT GAIN MORE BY MISREPORTING

THE OTHER BIDDERS CANNOT GAIN MORE

THE BASIC REASON IS THAT THE PAYMENTS DO NOT ON THE REPORTED TYPES

<u>EXAMPLE</u>:

$B = \{ b_1 = 10,\ b_2 = 10,\ b_3 = 9,\ b_4 = 8,\ b_5 = 7,\ b_6 = 4 \}$

$S = \{ s_1 = 1,\ s_2 = 3,\ s_3 = 4,\ s_4 = 6,\ s_5 = 7,\ s_6 = 7 \}$

①

$P_{b_1} = P_{b_2} = P_{b_3} = \cdots = b_5 = 7$

$P_{s_1} = P_{s_2} = \cdots = s_5 = 7$

$w_{b_1} = 10 - 7 = 3$ $\quad w_{b_4} = 8 - 7 = 1$ $\quad w_{s_2} = 7 - 3 = 4$

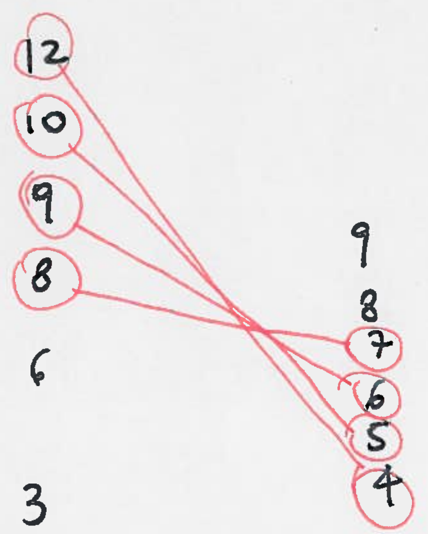
$w_{b_2} = 10 - 7 = 3$ $\quad w_{s_3} = 7 - 4 = 3$

$w_{b_3} = 9 - 7 = 2$ $\quad w_{s_1} = 7 - 1 = 6$ $\quad w_{s_4} = 7 - 6 = 1$

EXAMPLE:

$$B = \{ b_1 = 12,\ b_2 = 10,\ b_3 = 9,\ b_4 = 8,\ b_5 = 6,\ b_6 = 3 \}$$

$$S = \{ s_1 = 4,\ s_2 = 5,\ s_3 = 6,\ s_4 = 7,\ s_5 = 8,\ s_6 = 9 \}$$



$k = 4$

$b_5 = 6$, $s_5 = 8$

(1)

$\Rightarrow$ (b_1, s_1), (b_2, s_2), (b_3, s_3) MATCHED

$$P_{b_1} = P_{b_2} = P_{b_3} = 8$$

$$P_{s_1} = P_{s_2} = P_{s_3} = -7$$

$\Rightarrow$ TRADER'S REVENUE $= 3 = (8-7) * 3$

(2)

$$\frac{1}{2}(s_5 - b_5) = 7 \Rightarrow (b_1, s_1),\ (b_2, s_2),\ (b_3, s_3),\ (b_4, s_4)$$

$$P_{b_1} = P_{b_2} = P_{b_3} = P_{b_4} = 7$$

$$P_{s_1} = P_{s_2} = P_{s_3} = P_{s_4} = -7$$

REVENUE OF THE AUCTIONEER = $(7-7) \cdot 4 = 0$

2 $\frac{1}{2}(7+4) = 5.5$

$5.5 \notin [s_5, b_5] \Rightarrow$ 2 CANNOT BE APPLIED

DISPLAY ADVERTISEMENT

- USERS
- PUBLISHERS (WEB PAGES)
- ADVERTISERS

THERE ARE MANY WEB PAGES (PUBLISHER) AND MANY ADVERTISERS

THERE ARE INTERMEDIARIES:

- AD AGENCIES
- AD NETWORKS
- PUBLISHER NETWORKS

THEIR SERVICES ARE OVERLAPPING

ADVERTISER → AD AGENCY → AD NETWORK → Publisher ↔ USER

AD EXCHANGE: COMMON MARKET PLACE FOR ALL THE PLAYERS

- RIGHT MEDIA → 9 BILLION TRANSACTIONS PER DAY (100,000 PLAYERS)
- Ad ECN
- Double click (GOOGLE)

CPM (COST per MILLE / THOUSAND IMPRESSIONS)