COMBINATORIAL AUCTION

SCENARIO: THERE ARE:

- ITEMS $I = \{i_1, \dots i_n\}$
- BIDDERS
- EACH BID CAN BID OVER A BUNDLE $S \subseteq I$
- EACH BID CAN BE DIFFERENT (OR THE SAME BIDDER)

MODEL:

- ITEMS: $I = \{i_1, \dots i_n\}$
- BUNDLES: $S_T \subseteq I$
- BID/VALUES: $v_{a,T}$, value of bidder a over bundle T

QUESTION: WHY COMBINATORIAL? BECAUSE, GIVEN A SET OF ITEMS WE HAVE A COMBINATORIAL NUMBER OF POSSIBLE BUNDLES

EXAMPLE: $I = \{A, B, C\}$

VALUES:

Bundle	a_1	a_2	a_3	MAX
A	2	1	0	2
B	1	1	1	1
C	0	0	2	2
A B	3	3	2	3
A C	4	1	2	4
B C	2	2	5	5
A B C	5	6	7	7

PROBLEM: FINDING THE BEST ALLOCATION IS NP-HARD

$\longrightarrow$ NO POLYNOMIAL-TIME ALGORITHM

PROBLEM: DESIGNING THE FASTEST ALGORITHM RETURNING THE EXACT OPTIMAL ALLOCATION

THIS ALGORITHM WILL BE USED IN THE VCG

PROBLEM: DEFINING AN APPROXIMATE MECHANISM FINDING AN APPROXIMATE OPTIMAL ALLOCATION

THIS ALGORITHM CANNOT BE USED IN THE VCG

ALGORITHM: BRANCH-AND-BOUND WITH HEURISTICS AND PRUNING

HEURISTICS:

GIVEN THE MAXIMAL BIDS, FOR EACH ITEM i WE COMPUTE $h_i = \max_{S(i)} \left\{ \frac{\bar{v}_{S(i)}}{|S(i)|} \right\}$ WHERE:

- $S(i)$ is a generic bundle containing item i
- $|S(i)|$ is the size of $S(i)$
- $\bar{v}_{S(i)}$ is the best bid over $S(i)$

EXAMPLE:

$$h_A = \max \left\{ 2,\ 3/2,\ 4/2,\ 7/3 \right\} = 2.3$$

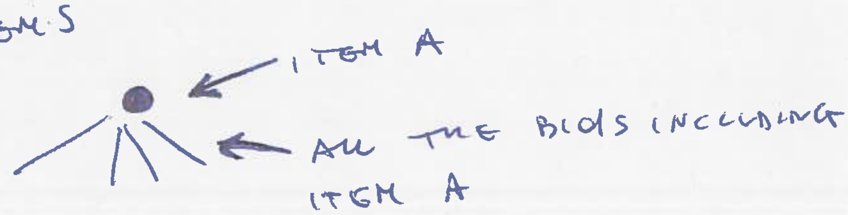
$$h_B = \max \left\{ 1,\ 3/2,\ 5/2,\ 7/3 \right\} = 2.5$$

$$h_C = \max \left\{ 2,\ 4/2,\ 5/2,\ 7/3 \right\} = 2.5$$

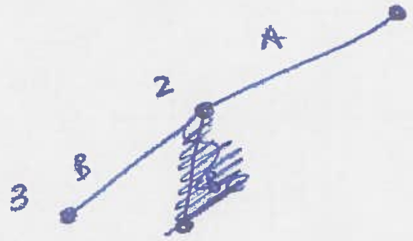
BRANCH-AND-BOUND:

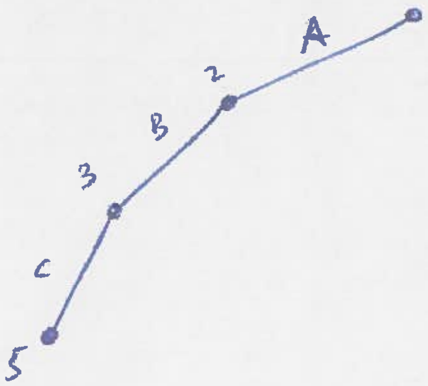
IF SOME ITEM HAS NO SINGLE ITEM BID, A DUMMY BID IS INTRODUCED

- BRANCH ON ITEMS



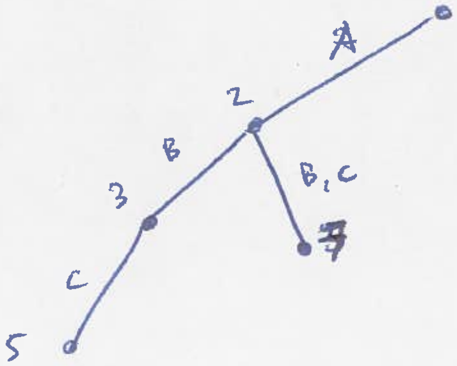
- DEPTH FIRST
- PRUNING IF THE CURRENT VALUE OF THE ALLOCATION + HEURISTICS IS NOT GREATER THAN THE BEST FOUND SOLUTION



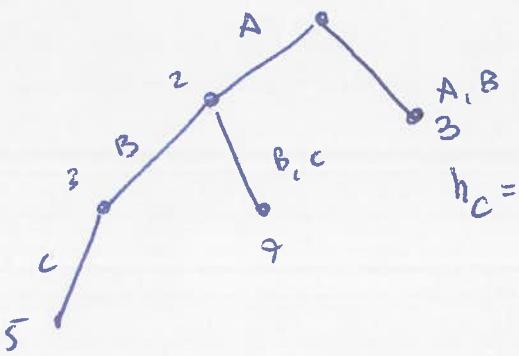


→ ALLOCATION WITH A VALUE OF 5

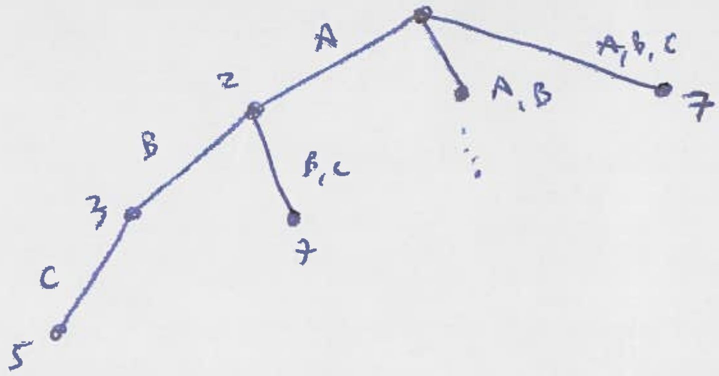
$\underline{v} = 5$



→ THE CURRENT OPTIMAL ALLOCATION IS $\boxed{A}$ $\boxed{B}$ $\boxed{C}$ WITH $\underline{v}=7$



$h_C = 2.5$ } ⇒ TOTAL $5.5 < \underline{v}$ ⇒ NO EXPANSION BEYOND $\boxed{A|B}$



EXCERCISE:

a	3
b	7
c	1
d	2
a, c	8
a, d	5
b, c	7
b, d	10
a, c, d	10
b, c, d	12

PROPERTY: BRANCH-ON-ITEMS ALGORITHM REQUIRES EXPONENTIAL TIME IN THE NUMBER OF ITEMS

PROPERTY: THE VCG REQUIRES EXPONENTIAL TIME GIVEN THAT THE OPTIMAL ALLOCATION CAN BE FOUND IN EXPONENTIAL TIME

EXERCISE: APPLY THE VCG TO THE ABOVE EXAMPLE

ASSUMPTION: Bidders ARE SINGLE-MINDED IF THEIR Bids ARE

- $v_{i,J} \neq 0$ ONLY FOR A SINGLE J

IN PRACTISE EACH Bidder has a positive (STRICT) VALUATION ONLY FOR A SINGLE BUNDLE

PROPERTY: EVEN WITH SINGLE-MINDED Bidders, FINDING THE OPTIMAL ALLOCATION IS NP-HARD

PROPERTY: APPROXIMATING THE OPTIMAL ALLOCATION WITHIN A FACTOR OF $\sqrt{m}$ IS POSSIBLE IN POLYNOMIAL TIME ($m = |I|$

ALGORITHM:

- ORDINA LE BID RISPETTO A $\frac{v_i}{\sqrt{|S_i|}}$
- $W \leftarrow \emptyset$
- SCAN THE BID, IF $|S_i| \cap W = \emptyset \rightarrow$ ADD S_i IN W

EXAMPLE:

$$\frac{v_A}{\sqrt{\boxed{A}}} = 2 \qquad \frac{v_B}{\sqrt{\boxed{B}}} = 1 \qquad \frac{v_C}{\sqrt{\boxed{C}}} = 2$$

$$\frac{N_{AB}}{\sqrt{\boxed{A|B}}} = \frac{3}{\sqrt{2}} \simeq 2.12 \qquad \frac{N_{AC}}{\sqrt{\boxed{A|C}}} = \frac{4}{\sqrt{2}} \simeq 2.82 \qquad \frac{N_{BC}}{\sqrt{\boxed{B|C}}} = \frac{5}{\sqrt{2}} \simeq 3.53$$

$$\frac{N_{ABC}}{\sqrt{\boxed{A|B|C}}} = \frac{7}{\sqrt{3}} \simeq 4.04$$

$\Rightarrow$ $\boxed{A|B|C}$, $\boxed{B|C}$, $\boxed{A|C}$, $\boxed{A|B}$, $\boxed{A}$, $\boxed{C}$, $\boxed{B}$

$\Rightarrow$ APPROXIMATE SOLUTION $\boxed{A|B|C}$, THE VALUE IS 7

THE GUARANTEE IS THAT OPTIMAL SOLUTION IS $\leq 7 \cdot 3$

OBSERVATION: THE VCG CANNOT BE ADOPTED, BUT THERE IS A MECHANISM THAT IS DST.

MECHANISM: THE PAYMENTS ARE:

FOR EACH $i \in W$, $p_i = N_T \sqrt{\frac{|S_i|}{|S_T|}}$

WHERE T IS THE SMALLEST INDEX SUCH THAT $S_i \cap S_T \neq \emptyset$ AND FOR ALL $k < T$, $k \neq i$ $S_k \cap S_T = \emptyset$ (IF NO SUCH T EXISTS THEN $p_i = 0$)