COMBINATORIAL AUCTION

There are $|I|$ items that can be sold to buyers. A buyer can make a bid on ANY bundle of such items. Items are indivisible.

$I = \{A, B, C\}$, bids on: A, B, C, AB, BC, AC, ABC

Thus, there are $2^{|I|} - 1$ bundles on which each agent can potentially bid.

As we can see, the input is potentially exponential in I.

The question is: is the complexity of this problem just related to the size of the input or the problem itself is difficult?

To answer, we restrict our attention to SINGLE-MINDED bidders, i.e., each bidder can bid on a single bundle.

$|I| = m = n$ items, $|N| = n = n$ bids: there exists a polynomial algorithm to search for the approximated allocation, i.e., the values of the bids that should be accepted.

$SW(x) = \sum_i x(B) \cdot v(B)$ $v(B)$ is the value of the bid B, i.e., θ_i
$x(B)$ is a binary variable: it is equal to 1 if B is accepted

Now we want to apply the VCG to find the allocation x^* that maximizes x.

CA with VCG with single-minded agents is NP-hard.

We reduce this problem from INDEPENDENT SET that is NP-hard.

Memory refresh:

- Decision problem: a question with a yes-no answer depending on the value of some input parameters.
- NP: it is the set of all decision problems such that for each instance with a yes-answer, there exists a concise certificate that allows to verify in polynomial time that the answer is yes.
- NP-hard problem: every problem in NP reduces to it in polynomial time.

$NP \dashrightarrow IS \dashrightarrow \Pi$

Independent Set (IS)

INPUT: a graph $G = (V, E)$
an integer K

QUESTION: is there a set of vertices $S \subseteq V$, $|S| \geq K$, such that $\forall s, s' \in S$, $(s, s') \notin E$?

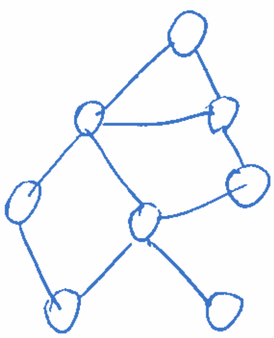
We are requiring that for every couple of vertices in S, there is no edge connecting the two.

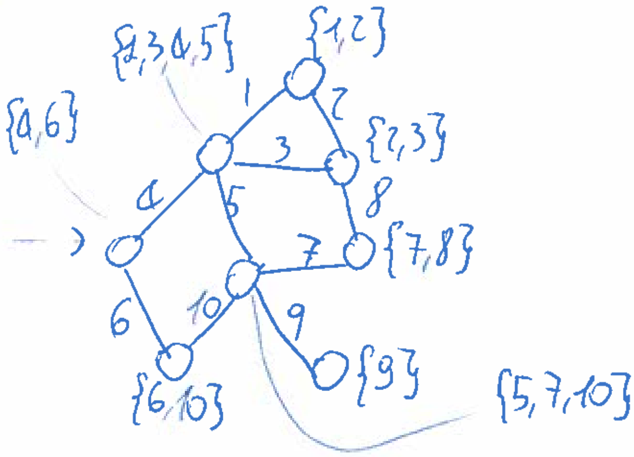
Combinatorial Auction (CA)

INPUT: $I = \{i_1, \dots, i_m\}$ = set of items
$B = \{b_1, \dots, b_n\}$ = set of bundles,
$b_i = 1$, an integer K'

QUESTION: is there an allocation whose values $v \geq K'$?

For every instance of IS, we build an instance of CA





edges → items
nodes → bundles
K → K'

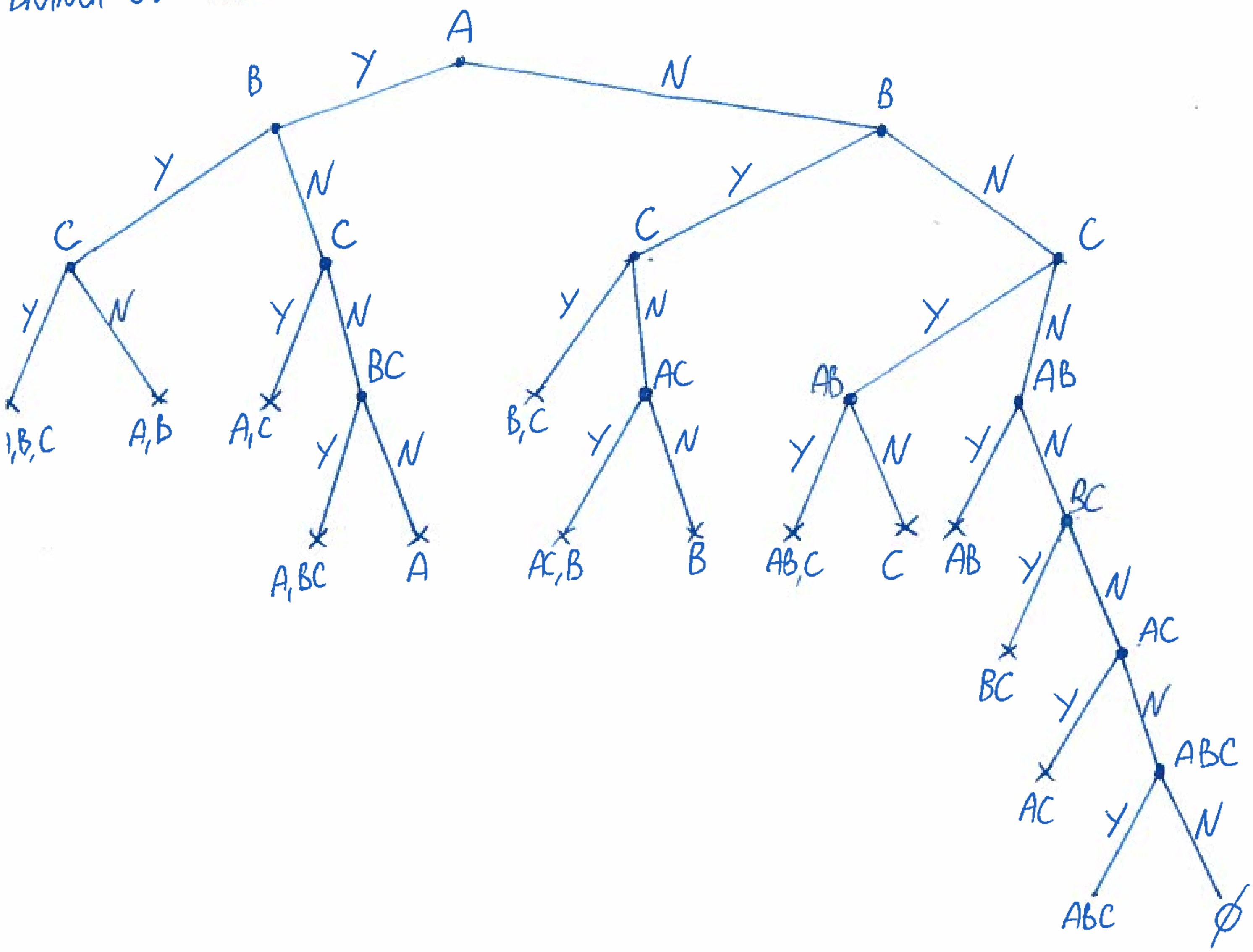
YES: there is a IS of cardinality $\geq K$ → Select the nodes corresponding to the IS. Each node ≡ bundle whose valu is 1. Since $K = K'$, this is a YES instance.

Select the nodes corresponding to the bundles. $K = K'$ so at least K nodes will be selected and, by construction, they form a IS (if not, this means the same item has been assigned to more bidders). Thus, this is a YES instance. ← YES: there is an allocation whose val is $\geq K'$

Actually, you can apply VCG, but you'll never do that in practice.

Let us see an example to apply a VCG algorithm to find the optimal allocation

BRANCH ON BUNDLES



This way we have x^*: which are the payments?

BUNDLES	b_1	b_2	b_3	MAX b_i
A				
B				
C				
A B				
B C				
A C				
A B C				

To compute $SW(x^*)$, we take the max b_i corresponding to the bundles of x^*

To compute $SW_{-i}(x^*_{-i})$, we remove column b_i and compute the new allocation using the tree

Payments are as usual:

$p_i = SW_{-i}(x^*_{-i}) - SW_{-i}(x^*)$