A note on comparative ambiguity aversion and justifiability

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Setup

- Standard decision problem under uncertainty.
- Action set A.
- State space S.
- Monetary consequence space $C \subseteq \mathbb{R}$.
- Consequence function (game form) $g : A \times S \to C$.
- vN-M utility function $v : C \to \mathbb{R}$.
- Payoff function $r = v \circ g : A \times S \to \mathbb{R}$, i.e.,

$$r(a, s) = v(g(a, s)).$$

Setup: smooth ambiguity

- DM posits a collection $\Sigma \subseteq \Delta(S)$ of possible stochastic models σ for states; e.g., S=set of colors of balls, Σ=set of possible urn compositions.
- Actions ranked by the smooth ambiguity criterion

$$V_{\phi,r}(a,\mu) = \phi^{-1}\left(\int_{\Sigma} \phi\left(\int_{S} r(a,s)\,\sigma(\mathrm{d}s)\right)\mu(\mathrm{d}\sigma)\right)$$

where ϕ is strictly increasing (and cont.) and μ is a subjective belief on Σ (Klibanoff, Marinacci, and Mukerji, 2005).

Setup: special cases

- Special case: when $\phi = \mathrm{Id}$ we have Savage (1954)'s SEU criterion

$$V_{\mathrm{Id},r}(a,\mu) = \int_\Sigma \int_S r(a,s)\,\sigma(\mathrm{d}s)\,\mu(\mathrm{d}\sigma) = \int_S r(a,s)\,\sigma_\mu(\mathrm{d}s)$$

where $\sigma_\mu(\cdot) = \int_\Sigma \sigma(\cdot)\,\mu(\mathrm{d}\sigma)$ is the predictive prob. induced by μ.

- Limit case: Gilboa and Schmeidler (1989)'s maxmin criterion

$$V_{\infty,r}(a,\mu) = \min_{\sigma \in \operatorname{supp}\mu} \int r(a,s)\,\sigma(\mathrm{d}s)\,.$$

Justifiability

Definition

The collection of justifiable actions for ambiguity attitudes ϕ and risk attitudes r, given Σ, is

$$\mathcal{J}_{\phi,r}(\Sigma) = \left\{a \in A : \exists \mu \in \Delta(\Sigma), \forall a' \in A,\ V_{\phi,r}(a,\mu) \geq V_{\phi,r}\left(a',\mu\right)\right\}.$$

- The collection of all actions that are best replies, according to $V_{\phi,r}$, to some belief μ over Σ.
- Old best-reply-to-some-belief concept, applied here to criterion $V_{\phi,r}$.

Risk setup

- In our monetary setting

$$r' = \psi \circ r = (\psi \circ v) \circ g$$

  with ψ strictly increasing (cont.) and concave, is the payoff function of a *more risk averse* DM.
- The next result was independently proved by Weinstein (2013).
- δ_s is the Dirac probability concentrated on state s.
- S is embedded in Σ, written $S \subseteq \Sigma$, if $\{\delta_s\}_{s \in S} \subseteq \Sigma$.

Comparative risk aversion

Theorem

Let S be compact and $S \subseteq \Sigma$. If $r' = \psi \circ r$ for some concave, continuous and strictly increasing function ψ, then

$$\mathcal{J}_{\mathrm{Id},r}(\Sigma) \subseteq \mathcal{J}_{\mathrm{Id},r'}(\Sigma).$$

- Higher risk aversion enlarges the collection of justifiable actions.

Risk example

- Consider the game form:

$g:$	s^1	s^2
t	0	1
m	$\frac{1}{3}$	$\frac{1}{3}$
b	1	0

- Suppose the DM is a SEU maximizer ($\phi = \mathrm{Id}$). If the DM is *risk neutral* ($r = g$), action m is unjustifiable: for every belief $\mu \in \Delta(\Sigma)$,

$$\begin{aligned} V_{\mathrm{Id},g}(m,\mu) &= \frac{1}{3} < \frac{1}{2} \leq \max\left\{V_{\mathrm{Id},g}(b,\mu), V_{\mathrm{Id},g}(t,\mu)\right\} \\ &= \max\left\{\sigma_\mu(s^1), 1-\sigma_\mu(s^1)\right\}. \end{aligned}$$

Example: risk neutrality

- If $S \subseteq \Sigma$, then $J_{\mathrm{Id},g}(\Sigma) = \{t, b\}$.
- Action b (resp., t) is a best reply to μ iff $\sigma_{\mu}(s^1) \geq 1/2$ (resp., $\sigma_{\mu}(s^1) \leq 1/2$).

Example: comparative risk aversion

- Suppose DM is risk averse, with $v_\theta(c) = c^{1/\theta}$ where $\theta \geq 1$ parametrizes risk aversion.
- The payoff function is $r_\theta = v_\theta \circ g$.
- It is easy to see that

$$\mathcal{J}_{\mathrm{Id}, r_\theta}(\Sigma) = \begin{cases} \{t, b\}, & \theta < \bar{\theta}, \\ \{t, m, b\}, & \theta \geq \bar{\theta}, \end{cases}$$

  where $\bar{\theta} = \log_2 3$ solves $(1/3)^{1/\theta} = 1/2$.
- The collection of justifiable actions thus enlarges as θ increases.

Ex.: comparative risk aversion, beliefs

- For $\theta \leq \bar{\theta}$, the set of beliefs justifying b (resp., t) is $\{\mu : \sigma_\mu(s^1) \geq 1/2\}$ (resp., $\{\mu : \sigma_\mu(s^1) \leq 1/2\}$); it shrinks as θ increases above $\bar{\theta}$, but it always contains δ_{s^1} (resp., δ_{s^2}).
- For any fixed belief μ, "risky" actions become less appealing as risk aversion increases, hence they may get out of $\arg\max_a V_{\mathrm{Id},\psi\circ r}(a, \mu)$. But if they are justifiable for low risk aversion, then they are always justified by "extreme beliefs".

Comp. risk aversion, simple proof

Easy case: $S = \Sigma$ *finite*, identification $\delta_s = s$, thus

$$V_{\mathrm{Id},r}(a, \mu) = \sum_s r(a, s)\mu(s).$$

Lemma

(Wald-Pearce, WP) *justifiable* $\Longleftrightarrow$ *undominated by mixed, that is,*

$$\mathcal{J}_{\mathrm{Id},r}(S) = \left\{ a \in A : \forall \alpha \in \Delta(A), \exists s \in S, r(a, s) \geq \sum_{a' \in A} \alpha(a') r(a', s) \right\}$$

Let ψ be str.increasing, cont., concave. Show $\mathcal{J}_{\mathrm{Id},r}(S) \subseteq \mathcal{J}_{\mathrm{Id},\psi \circ r}(S)$.

For each $a \in \mathcal{J}_{\mathrm{Id},r}(S)$, by $(\mathrm{WP} \Rightarrow)$

$$\begin{aligned}\forall \alpha \quad &\in \quad \Delta(A), \exists s \in S,\ r(a,s) \geq \sum_{a' \in A} \alpha(a') r(a',s) \\ &\overset{\text{(Jensen)}}{\geq} \quad \psi^{-1}\left(\sum_{a' \in A} \alpha(a') \psi\left(r(a',s)\right)\right)\end{aligned}$$

$$\forall \alpha \in \Delta(A), \exists s \in S,\ \psi\left(r(a,s)\right) \geq \sum_{a' \in A} \alpha(a') \psi\left(r(a',s)\right)$$

i.e., a is undominated for $\psi \circ r$, hence $(\mathrm{WP} \Leftarrow)$ a is justifiable for $\psi \circ r$. Thus, $\mathcal{J}_{\mathrm{Id},r}(S) \subseteq \mathcal{J}_{\mathrm{Id},\psi \circ r}(S)$. ■

Comparative ambiguity aversion

Theorem

Let Σ be compact. If $\phi' = \varphi \circ \phi$ for some concave, continuous, and strictly increasing function φ, then

$$\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\phi',r}(\Sigma).$$

- Higher ambiguity aversion enlarges the collection of justifiable actions.

Ambiguity: example

- Consider again the game form:

$g:$	s^1	s^2
t	0	1
m	$\frac{1}{3}$	$\frac{1}{3}$
b	1	0

- Suppose the DM is *risk neutral* $(r = g)$ and that $S \subseteq \Sigma$, e.g., $\Sigma = \{\delta_{s^1}, \delta_{s^2}\}$.

Comparative ambiguity: example

- Let $\phi_\theta(x) = x^{1/\theta}$, where $\theta \geq 1$ parametrizes ambiguity aversion.
- It can be shown (nontrivial) that

$$\mathcal{J}_{\phi_\theta, r}(\Sigma) = \begin{cases} \{t, b\}, & \theta < \bar{\theta}, \\ \{t, m, b\}, & \theta \geq \bar{\theta}, \end{cases}$$

  where $\bar{\theta} = \log_2 3$.
- The collection of justifiable actions thus enlarges as θ increases.

Comparative ambiguity: example, beliefs

- The *set of beliefs justifying* b (resp., t) contains $\{\mu \in \Delta(\{\delta_{s^1}, \delta_{s^2}\}) : \mu(\delta_{s^1}) \in [1/2, 1]\}$ (resp., $\{\mu \in \Delta(\{\delta_{s^1}, \delta_{s^2}\}) : \mu(\delta_{s^1}) \in [0, 1/2]\}$) for $\theta \leq \bar{\theta}$; it *shrinks* as θ increases above $\bar{\theta}$, but it always contains μ: $\mu(\delta_{s^1}) = 1$ (resp., μ: $\mu(\delta_{s^2}) = 1$).
- For any fixed belief μ, ambiguous actions (actions that do not "hedge" against uncertainty) become less appealing as ambiguity aversion increases, hence they may get out of $\arg\max_a V_{\phi,r}(a, \mu)$ as $-\phi''/\phi' \uparrow$. But if they are justifiable for low $-\phi''/\phi'$, then they are always justified by "extreme beliefs".

Ambiguity and risk

- Summing up, higher aversion to either risk or ambiguity enlarges the collection of justifiable actions.
- It is a purely comparative result that does not require either risk or ambiguity aversion (i.e., the functions r and ϕ are not assumed to be concave).
- The proof is based on an *abstract version of the duality lemma* of Pearce (1984), a version of the classic Complete Class Theorem of Wald (see, e.g., Wald, 1949).

Comp. ambiguity aversion, simple proof

Easy case: Σ *finite*, let

$$R(a, \sigma) = \int_S r(a, s)\sigma(\mathrm{d}s)$$

then

$$V_{\phi,r}(a, \mu) = \phi^{-1}\left(\sum_{\sigma\in\Sigma} \phi\left(R(a, \sigma)\right)\mu(\sigma)\right).$$

Since ϕ^{-1} is str. increasing,

$$\mathcal{J}_{\phi,r}(\Sigma) = \bigcup_{\mu\in\Delta(\Sigma)} \arg\max_{a\in A} \sum_{\sigma\in\Sigma} \phi\left(R(a, \sigma)\right)\mu(\sigma)$$

(note affinity in μ).

Apply WP lemma to $\phi \circ R : A \times \Sigma \to \mathbb{R}$, then $\mathcal{J}_{\phi,r}(\Sigma) =$

$$\left\{ a \in A : \forall \alpha \in \Delta(A), \exists \sigma \in \Sigma, \phi\left(R(a,\sigma)\right) \geq \sum_{a' \in A} \alpha(a')\phi\left(R(a',\sigma)\right) \right\}.$$

The proof is analogous to the case of risk aversion (analogies $\varphi : \psi$, $\Sigma : S$, $\phi \circ R : r$).

Let φ be str. increasing, cont. and concave. Show $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\varphi \circ \phi, r}(\Sigma)$.

For each $a \in \mathcal{J}_{\phi,r}(\Sigma)$, by $(\mathrm{WP} \Rightarrow)$

$$\begin{aligned}\forall \alpha \ \in \ \Delta(A), \exists \sigma \in \Sigma,\ \phi\left(R(a,\sigma)\right) &\geq \sum_{a' \in A} \alpha(a')\phi\left(R(a',\sigma)\right) \\ &\overset{\text{(Jensen)}}{\geq} \varphi^{-1}\left(\sum_{a' \in A} \alpha(a')\varphi\left(\phi\left(R(a',s)\right)\right)\right)\end{aligned}$$

$$\forall \alpha \in \Delta(A), \exists \sigma \in \Sigma,\ \varphi\left(\phi\left(R(a,\sigma)\right)\right) \geq \sum_{a' \in A} \alpha(a')\varphi\left(\phi\left(R(a',\sigma)\right)\right)$$

i.e., a is undominated for $\varphi \circ \phi$, hence by $(\mathrm{WP} \Leftarrow)$ a is justifiable for $\varphi \circ \phi$. Thus, $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\varphi\circ\phi,r}(\Sigma)$. ■

Extensions: rationalizability

- Justifiability in decision problems can be seen as a special case of rationalizability in games. The previous analysis has implications for rationalizability:
- Under complete information (which implies common knowledge of risk and ambiguity attitudes) higher risk or ambiguity aversion enlarges the set of rationalizable outcomes.
- The same result holds if what is common knowledge is an upper bound on risk, or ambiguity aversion and we increase the upper bound.

Unbounded ambiguity aversion

- Consider the ambiguity aversion limit case as $-\phi'/\phi'' \uparrow +\infty$ pointwise.
- As well known, $V_{\phi,r}(a,\mu)$ tends to the maxmin criterion

$$V_{\infty,r}(a,\mu) = \min_{\sigma \in \operatorname{supp} \mu} \int r(a,s)\,\sigma(\mathrm{d}s).$$

- Here $\mathcal{J}_{\infty,r}(\Sigma)$ is the set of justifibale actions under infinite ambiguity aversion (maxmin criterion)

$$\left\{a \in A : \exists \mu \in \Delta(\Sigma), \forall a' \in A,\ V_{\infty,r}(a,\mu) \geq V_{\infty,r}(a',\mu)\right\}.$$

Unbounded ambiguity aversion

- We have $\mathcal{J}_{\phi,r}(\Sigma) \subseteq \mathcal{J}_{\infty,r}(\Sigma)$ (upper hemicontinutiy w.r.t. ambiguity aversion).
- Given our monotonicity result (Theor. 4), the limit set of $\mathcal{J}_{\phi,r}(\Sigma)$ as $-\phi'/\phi'' \uparrow +\infty$ can be characterized via a dominance notion due to Boergers (1993) applied here to moldes σ rather than states:

$$\lim_{-\phi'/\phi''\uparrow+\infty} \mathcal{J}_{\phi,r}(\Sigma) \overset{\text{(mon.)}}{=} \bigcup_{\phi \text{ str.incr,conc.}} \mathcal{J}_{\phi,r}(\Sigma) \overset{\text{(Boergers)}}{=}$$

$$\left\{ \begin{array}{c} a \in A : \exists\hat{\Sigma} \subseteq \Sigma, \forall a' \in A, \\ \left(\exists\sigma \in \hat{\Sigma}, R(a,\sigma) > R(a',\sigma)\right) \vee \left(\forall\sigma \in \hat{\Sigma}, R(a,\sigma) \geq R(a',\sigma)\right) \end{array} \right\}$$

set of pure actions that are not "weakly dominated" by other pure actions within at least one subset $\hat{\Sigma}$ of models.
- Actions that are justifiable under maxmin may not belong to such limit set, i.e., in general

Unbounded ambiguity aversion: example

- Given $0 \leq \varepsilon < 1$, consider the payoff function (in utils):

$r:$	s^1	s^2
t	1	0
m	0	1
b	ε	ε

- $b \in \mathcal{J}_{\infty,r}(\Sigma)$ for each $\varepsilon \geq 0$.
- If $\varepsilon > 0$, there exists ϕ str. increasing, concave such that $b \in \mathcal{J}_{\phi,r}(\Sigma)$.
- If $\varepsilon = 0$, there is no such ϕ.

Risk vs. ambiguity: a richer setup

- Enrich the setup through randomized consequences $g : A \times S \to \Delta_0 (C)$.
- The payoff function becomes

$$r(a, s) = \sum_{c \in C} v(c) \, g(a, s)(c)$$

where $v : \Delta_0(C) \to \mathbb{R}$.

- It is easy to see that the *previous comparative-ambiguity result still holds.*

Risk vs. ambiguity: example

- Suppose $C = \{0, 1\}$ and consider:

$g:$	s^1	s^2
t	δ_1	δ_0
m	$\frac{2}{3}\delta_0 + \frac{1}{3}\delta_1$	$\frac{2}{3}\delta_0 + \frac{1}{3}\delta_1$
b	δ_0	δ_1

- Risk attitudes do not matter.
- Set $v(0) = 0$ and $v(1) = 1$, so that r is given by

$r:$	s^1	s^2
t	1	0
m	$\frac{1}{3}$	$\frac{1}{3}$
b	0	1

Risk vs. ambiguity: example

- Let $\phi_\theta(x) = x^{1/\theta}$, with $\theta \geq 1$. As before, it can be shown

$$\mathcal{J}_{\phi_\theta, r}(\Sigma) = \begin{cases} \{t, b\}, & \theta < \bar{\theta}, \\ \{t, m, b\}, & \theta \geq \bar{\theta}, \end{cases}$$

where $\bar{\theta} = \log_2 3$. But here *risk* attitudes are *irrelevant*

Discussion: self-confirming equilibrium

- In Battigalli et al. (2015) we show that higher ambiguity aversion enlarges the set of self-confirming equilibria, which represent the possible steady states of recurrent play given that agents learn from personal experience.
- The result is analogous, but the similarity is only superficial and different assumptions are used.

Discussion: discipline on beliefs

- The theorem in Battigalli et al. (2015) is an equilibrium phenomenon, the result of a strong discipline on beliefs that self-confirming equilibria impose by requiring their consistency with long-run feedback data, given the observability of consequences.
- Here instead beliefs are altogether unrestricted in decision problems and, in a game theoretic perspective, the present results are relevant for non-equilibrium analysis where the only restrictions on beliefs come from ex ante strategic reasoning (as the previous remarks on rationalizability show). The ex post observability of consequences is irrelevant for the analysis.

Discussion: comparative justifying beliefs

- Self-confirming actions are perceived as unambiguous best replies by players, whereas unused alternatives are typically perceived as ambiguous.
- Therefore, holding beliefs fixed, an increase in ambiguity aversion cannot change the best-reply status of self-confirming actions, and the set of confirmed beliefs justifying self-confirming actions *expands*.
- In sharp contrast, the set of beliefs justifying actions with uncertain concequences typically *shrinks* as ambiguity aversion increases (see actions b, t in example).

Abstract Pearce lemma: setup

- As mentioned, the analysis is based on an abstract version of the duality lemma of Pearce (1984).
- The original game theoretic version is based on randomization.
- In the abstract, *convex subsets* of a *vector space*.
- Fix two nonempty subsets A_1 and A_2 of a Hausdorff locally convex topological vector space.
- Let $B_i = \mathrm{co}\, A_i$ and $\bar{B}_i = \overline{\mathrm{co}} A_i$ denote respectively the convex hull of A_i and its closure.

Abstract Pearce lemma: dominance

- Given $F : B_1 \times \bar{B}_2 \longrightarrow \mathbb{R}$, we say that $a_1^* \in A_1$ is *dominated* if

$$\exists a_1 \in A_1, \forall a_2 \in A_2, \qquad F(a_1^*, a_2) < F(a_1, a_2),$$

  otherwise we say that a_1^* is *undominated*
- a_1^* is *co-dominated* if

$$\exists b_1 \in B_1, \forall a_2 \in A_2, \qquad F(a_1^*, a_2) < F(b_1, a_2),$$

  otherwise we say that a_1^* is *co-undominated.*

Abstract Pearce lemma

Theorem

Suppose that:
(i) A_2 *is closed and* $\bar{B}_2$ *is compact,*
(ii) F *is quasiconcave and upper semicontinuous on* B_1*,*
(iii) F *is affine and continuous on* $\bar{B}_2$*.*
An element $a_1^* \in A_1$ *is co-undominated only if there exists some* $b_2 \in \bar{B}_2$ *such that* $a_1^* \in \arg\max_{a_1 \in A_1} F(a_1, b_2)$*. The converse is true if* F *is affine on* B_1*.*

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