# Tactical vote in committees with applications to law, teaching, business and sports

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## Example (Law)

A juror has two choices–convict or acquit–but the outcome of the jury as a whole has a third option: a hung jury. For purposes of the example, we will suppose there are 5 jurors and each will vote for either conviction or acquittal and the outcome of the vote will be the unanimous decision of the jury, with a hung jury (and hence a potential retrial) if unanimity is not achieved.

$$V(A, C) = \begin{cases} \text{acquit}, & \text{if } |A| = 5 \\ \text{convict}, & \text{if } |C| = 5 \\ \text{hung jury}, & \text{otherwise} \end{cases} \qquad |A| + |C| = 5$$

- honest assessment of the evidence
- strategic vote

## Example (Teaching)

At a certain school each student is given a grade in each class of A, B, C, D or F. A student

- graduates with honors if he passes every course (with a B or A) and has the numbers of Bs is at most two.
- graduates if he passes every course (with a D or better) and has a grade point average of a C average or better.
- not graduated otherwise.

Each of his teachers can then be thought of as voters deciding if he should graduate. A teacher might think that he deserves a *D* in her class but also wants him to graduate without honors. She would like to give him the lowest grade possible without costing him his graduation.

## Example (Teaching)

- $V$: is the voting rule
- $n$: is the number of graded subjects, (e.g., equally weighted)
- $j = 5$: is the number of allowable grades per subject: A, B, C, D, F (equivalent to: 5, 4, 3, 2, 1)
- $k = 3$: is the number of allowable global grades: honors, graduate without honors, not graduated
- Notation: $G_A$ is the set of subjects with an $A$ grade for the student.

$$V(G_A, G_B, G_C, G_D, G_F) = \begin{cases} \text{graduate with honors,} & \text{if } G_A \cup G_B = N,\ |G_B| \le 2 \\ \text{graduate,} & \text{if } F = \emptyset,\ 2|G_A| + |G_B| \ge |G_D| \\ \text{not graduated,} & \text{otherwise} \end{cases}$$

## Example (Business)

Suppose each member of a budget committee is asked to vote to allocate either 5, 10, 15 or 20 percent of the budget to fund a project. The final allotment will be the average or median of the members' suggestions.

Again, if a committee member is more interested in the outcome of the averaging than her recommendation corresponding to her honest assessment, then she may wish to adjust her recommendation based on what she anticipates will be the recommendation of the others.

Only when she wants a middle value, does she have to consider strategizing.

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- In all of the examples the voting has been symmetric or anonymous.
- The framework of $(j, k)$-games is certainly flexible to allow for the voters to have different roles.

## Example (More business)

Suppose in the above example there is a budgetary supervisor who determines the maximum amount that can be spent on a project. Hence, he also votes $5, 10, 15$ or 20 percent, but his vote represents the maximum amount that can be spent on the project. Thus the outcome of the voting will be the smaller of the average of all the non-supervisors' votes and the vote of the supervisor.

$$V(I_{20}, I_{15}, I_{10}, I_5) = \begin{cases} \text{average of non-sup. votes,} & \text{if } S \in I_{20} \\ \min\{15, \text{average of non-sup. votes}\} & \text{if } S \in I_{15} \\ \min\{10, \text{average of non-sup. votes}\} & \text{if } S \in I_{10} \\ 5, & \text{if } S \in I_5 \end{cases}$$

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We can see that all of the non-supervisors could have an incentive to vote strategically, but it is not clear the supervisor does.

## Definition

A $(j, k)$-*(simple) game* consists of

- a finite set $N$ of voters,
- a set of input approvals $\{1, 2, \ldots, j\}$ ordered by $1 > 2 > \cdots > j$,
- a set of voting outcomes $\{v_1, v_2, \ldots, v_k\}$ ordered by $v_1 > v_2 > \ldots > v_k$,
- and a value function $V : j^N \to \{v_1, v_2, \ldots, v_k\}$ that is
  - monotonic, if for $X$, $Y$, if $X \,{}^{j}\!\subseteq Y$ then $V(X) \leq V(Y)$, and
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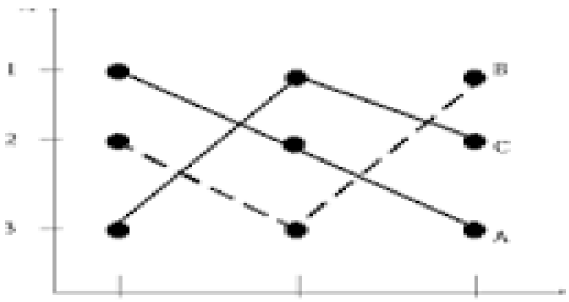
*Monotonicity for the business example with supervisor*, $N=\{a,b,S\}$,
Input set: $j=4$ ordered alternatives $20>15>10>5$,
Output set, $k=7$: $20>17.5>15>12.5>10>7.5>5$

$$(a,\emptyset,S,b)\,{}^j\!\subseteq(a,S,\emptyset,b)\,{}^j\!\subseteq(a,S,b,\emptyset)\,{}^j\!\subseteq(a,\{S,b\},\emptyset,\emptyset)$$

$$V(a,\emptyset,S,b)\le V(a,S,\emptyset,b)\le V(a,S,b,\emptyset)\le V(a,\{S,b\},\emptyset,\emptyset)$$

## Definition

A *preference function* for a player $a$ in a $(j, k)$-game, $V$, is a bijective function $P_a : \{v_1, v_2, \ldots, v_k\} \to \{1, 2, \ldots, k\}$ such that if $v_{i_1}$ and $v_{i_3}$ are such that $P_a(v_{i_1}) < P_a(v_{i_3})$ and $v_{i_2}$ is between $v_{i_1}$ and $v_{i_3}$ then $P_a(v_{i_2}) < P_a(v_{i_3})$. $P$ is said to be a universal preference function if $P(a)$ (written $P_a$) is a preference function for each $a \in N$. A $(j, k)$*-game with preferences* is a $(j, k)$-game, $V$ together with a universal preference function $P$. It is denoted $V_P$ or just $V$ when the preference function is clear. For each $a \in N$ we let $v_a^* = P_a^{-1}(1)$ be $a$'s most preferred outcome.



## Definition

In a $(j,k)$-game with preferences a player $a$ is said to have a *universally optimal vote* of level $i$ if whenever ordered $j$-partitions $A$ and $B$ agree outside of $a$ and $T_A(a) = i$ then $P_a(V(A)) \leq P_a(V(B))$. The game is said to be *manipulable by a* if $a$ has no universally optimal vote.

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A $(j,k)$-game with preferences is said to be *manipulable* if it is manipulable by some player.

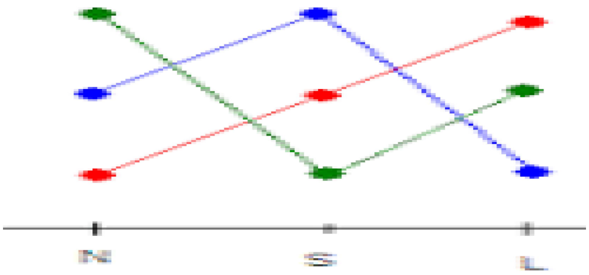
## Example (Law example revisited)

Judges: $a, b$: $A > H > C$
Judges: $c, d, e$: $C > H > A$
Judges: $a, b$: have $A$ as a universally optimal vote
Judges: $c, d, e$: have $C$ as a universally optimal vote.
Consequently, $V$ is not manipulable and therefore: $(ab, cde)$ is an NE.

Observe, whenever a $(j,k)$-game with preferences is not manipulable, it has an NE for which each voter chooses her universally optimal vote.

An unrestricted preference function is also called an agenda.

Preferences of three players
over 3 ordered output.




### Definition

Let $V$ be a $(j,k)$-game. $P$ is said to be an unrestricted universal preference function (or an agenda) if for each $a \in N$ , $P(a)$ (written $P_a$) is a bijective function $P_a : \{v_1, v_2, \ldots, v_k\} \to \{1, 2, \ldots, k\}$. A $(j,k)$-game with unrestricted preferences is a $(j,k)$-game, $V$ together with a universal unrestricted preference function $P$.

### Theorem (Gibbard theorem, 1973)

*Suppose $k > 2$ and let $\nu$ be a voting rule not containing a dictator. Then there exists a universal unrestricted preference function $P$ so that $\nu_P$ is manipulable.*

| Number of Voters: | Alessandro (A), Beppe (B), Daniela (D) |
|---|---|
| 5 | $B > A > D$ |
| 4 | $D > B > A$ |
| 2 | $A > D > B$ |

Plurality with honest vote: B wins; with tactical vote: D wins

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## Theorem (Gibbard-like theorem)

*Suppose $k > 2$ and let $V$ be a $(j, k)$-game with preferences not containing a dictator. Then there exists a universal unrestricted preference function $P$ so that $V_P$ is manipulable.*

## Example (Law example with two judges)

Outputs: acquit, hung jury, convict:

$$V(A, C) = \begin{cases} v_1 = \text{acquit}, & \text{if } |A| = 2 \\ v_3 = \text{convict}, & \text{if } |C| = 2 \\ v_2 = \text{hung jury}, & \text{otherwise} \end{cases}$$

Unrestricted preferences:
Judge $a$: $H > A > C$
Judge $b$: $C > A > H$
Neither $a$ nor $b$ have a universally optimal vote, thus the game is manipulable. Additionally, it has not an $NE$:

$$\begin{array}{ccc} (ab, \emptyset) & \stackrel{a}{\to} & (b, a) \\ (a, b) & \stackrel{b}{\to} & (ab, \emptyset) \\ (b, a) & \stackrel{b}{\to} & (\emptyset, ab) \\ (\emptyset, ab) & \stackrel{a}{\to} & (a, b) \end{array}$$

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Single-peaked preferences:
Judge $a$: $H > A > C$
Judge $b$: $C > H > A$
$b$ has a universally optimal vote, at $C$ input level, but $a$ does not have a universally optimal vote. Thus the game is manipulable. Additionally,

$$(a, b) \text{ is an NE}$$

## Example (Teaching example with five courses)

Outputs:
graduated with honors $>$ graduated without honors $>$ not graduated:

$$V(G_A, G_B, G_C, G_D, G_F) = \begin{cases} \text{graduate with honors,} & \text{if } G_A \cup G_B = N,\ |G_B| \leq 2 \\ \text{graduate,} & \text{if } F = \emptyset,\ 2|G_A| + |G_B| \geq |G_D| \\ \text{not graduated,} & \text{otherwise} \end{cases}$$

Single-peaked preferences:
Voter $a, b$: $v_1 > v_2 > v_3$. $a, b$ have a universally optimal vote, at $A$ input level,
Voter $c, d, e$: $v_2 > v_1 > v_3$. $c, d, e$ have not a universally optimal vote at any input level.

Thus, this game with preferences is manipulable.

$(ab, \emptyset, cde, \emptyset, \emptyset)$ is an NE

Positive results:

**Theorem**

- *Every* $(j, 3)$*-game with preferences has an NE.*

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Positive results:

**Theorem**

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- *Every two-player* $(j, k)$*-game with preferences has an NE.*
- *Every anonymous* $(2, k)$*-game with preferences has an NE.*

Negative result:

**Example**

For 3 voters one may find a $(3, 27)$-game with preferences without an NE.

**Conjecture**

Every anonymous $(j, k)$-game with preferences has an NE.

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**Question**

Find the minimum $k$ for which every $(3, k)$-game with preferences has an NE?
Such a number must fulfill $3 \leq k < 27$.

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- A manipulable $(j, k)$-game with preferences may have or not an NE.
- A manipulable but anonymous $(j, k)$-game with preferences has an NE. (Conjecture).